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SOME RESULTS ON BOX SPLINES

BY WOLFGANG DAHMEN AND CHARLES A. MICCHELLI

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Introduction. The purpose of this note is to describe progress made in understanding the box spline. This function is one of several polyhedral splines intensively studied during the past few years; see [1] for a survey of this subject.

For any set $X = \{x^1, \ldots, x^n\} \subseteq R^s \backslash \{0\}$ with $\langle X \rangle :=$ linear span of $X = R^s$, the box spline is defined by requiring that

$$\int_{R^s} B(x \mid X) f(x)\, dx = \int_0^1 \cdots \int_0^1 f\left(\sum_{i=1}^n t_i x^i\right) dt_1 \cdots dt_n$$

holds for all continuous functions on R^s. $B(x|X)$ is a smooth piecewise polynomial of degree $n - s$ and continuity class $C^{d(X)-1}(R^s)$, where

$$d(X) = \max\{m \colon \langle X \backslash Y \rangle = R^s,\ \forall Y \subset X \ni |Y| = m\}.$$

We are interested in the spline space spanned by integer translates of the box spline

$$S(X) = \langle \{B(\circ - \alpha \mid X) : \alpha \in Z^s\} \rangle.$$

It is important to know for the purpose of approximating smooth functions by scaled translates of box splines what polynomials are in $S(X)$. Denoting by $\Pi(R^s)$ the set of all polynomials on R^s, and by $\mathcal{D}'(R^s)$ the space of Schwartz distributions on $C_0^\infty(R^s)$, it is known that for $X \subset Z^s$,

$$S(X) \cap \Pi(R^s) = D(X),$$

where

$$D(X) = \{f \in \mathcal{D}'(R^s) : D_Y f = 0,\ \forall Y \subset X \ni \langle X \backslash Y \rangle \neq R^s\}$$

and $D_Y = \prod_{y \in Y} D_y$, D_y being the directional derivative in the direction of y.

The Nullstellensatz can be used to show that $\dim D(X) < \infty$ and $D(X) \subset \Pi(R^s)$. This fact and the others mentioned above, as well as relevant references, appear in [1].

Theorems. Our first result is

THEOREM 1. *For any $X \subset R^s \backslash \{0\}$ with $\langle X \rangle = R^s$, $|X| < \infty$, one has*

$$\dim D(X) = |\mathcal{B}(X)|,$$

where $\mathcal{B}(X) = \{Y : Y \subset X, |Y| = s, \langle Y \rangle = R^s\}$.

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