BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 3, November 1983

# INVARIANT THEORY OF $G_2$

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**Introduction.** Let $V$ denote $\mathbf{C}^n$, and let $G \subseteq \mathrm{SL}(V)$ be a classical subgroup. Then Classical Invariant Theory (CIT) describes the generators and relations of the algebra of invariant polynomial functions $\mathbf{C}[mV]^G$, where $m \in \mathbf{Z}^+$ and $mV$ denotes the direct sum of $m$ copies of $V$. Using the symbolic method (see [**7**]), one can then obtain a handle on the invariants of arbitrary representations of $G$. These classical methods and results have been very useful in many areas of mathematics.

Let $G$ be a connected, simple, and simply connected complex algebraic group. Then $G$ is classical except when $G = \mathrm{Spin}_n$, $n \geq 7$, or in case $G$ is an exceptional group $G_2$, $F_4$, $E_6$, $E_7$, or $E_8$. It would be useful to have an analogue of CIT for nonclassical $G$. We have succeeded in establishing an analogue for $G_2$ (described below). We also have a conjectured analogue for $\mathrm{Spin}_7$, but a complete proof requires a computation we are as yet unable to perform.

**The Cayley algebra, $G_2$, and the Main Theorem.** Let Cay denote the usual (complex) Cayley algebra (see [**3**]). Then Cay is a nonassociative, noncommutative algebra of dimension 8 over $\mathbf{C}$. Let Cay′ denote the (7-dimensional) span of all commutators of elements of Cay. Let $\mathrm{tr}: \mathrm{Cay} \to \mathbf{C}$ denote the linear map with kernel Cay′ which sends $1 \in \mathrm{Cay}$ to $1 \in \mathbf{C}$. Define $\bar{x} = -x + 2\,\mathrm{tr}(x) \cdot 1$, $x \in \mathrm{Cay}$. Then $x \mapsto \bar{x}$ is an involution such that $x\bar{x} = n(x) \cdot 1 \in \mathbf{C} \cdot 1$ for all $x \in \mathrm{Cay}$. Moreover,

$$x(xy) = x^2 y; \quad (yx)x = yx^2, \qquad x, y \in \mathrm{Cay}. \tag{1}$$

$$x^2 - 2\,\mathrm{tr}(x)x + n(x) \cdot 1 = 0, \qquad x \in \mathrm{Cay}. \tag{2}$$

(3) $x \mapsto n(x)$ is a nondegenerate quadratic form on Cay.

The identities in (1), called the alternative laws, are a weak form of associativity. Equation (2) is called the standard quadratic identity.

$G_2$ is the group of algebra automorphisms of Cay. Thus $G_2$ acts trivially on $\mathbf{C} \cdot 1$ and faithfully (and orthogonally) on Cay′. From now on, let $G$ denote $G_2$ and let $V$ denote Cay′. By (3), $V$ is $G$-isomorphic to its dual $V^*$.

The following is our main result.

Received by the editors April 25, 1983.
1980 *Mathematics Subject Classification*. Primary 17A36, 20F29, 20G05.
[1]Research partially supported by NSF grant MCS 7902937.


0273-0979/83 $1.00 + $.25 per page