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A GENERALIZATION OF TWO CLASSICAL CONVERGENCE TESTS FOR FOURIER SERIES, AND SOME NEW BANACH SPACES OF FUNCTIONS

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ABSTRACT. The norms of these spaces fill the gap between the uniform and the variation norms. Their duals are described in terms of generalized variation. One application of these spaces is a new convergence test for Fourier series which includes both the Dirichlet-Jordan and the Dini-Lipschitz tests [1].

1. The κ-entropy. $\kappa(s)$ will always denote a nondecreasing concave function on $[0,1]$ such that $\kappa(0) = 0$, $\kappa(1) = 1$; this implies that $\kappa(s)$ is continuous except, perhaps, at $s = 0$.

DEFINITION. Let $E = \{x_1 < x_2 < \cdots < x_n\} \subset [a,b]$ be a finite nonempty set. The following quantity will be called the *κ-entropy* of E (relative to $[a,b]$):

$$\kappa(E) = \kappa(E;[a,b]) = \sum_1^{n+1} \kappa((x_j - x_{j-1})/(b-a)), \tag{1}$$

where $x_0 = a$, $x_{n+1} = b$. For an arbitrary closed set $F \subset [a,b]$ we set

$$\kappa(F) = \kappa(F;[a,b]) = \sup\{\kappa(E)\colon E \subset F \text{ finite}\}. \tag{2}$$

Finally, we set $\kappa(\varnothing) = 0$.

The following properties of the κ-entropy are easily derived.

(i) $F_1 \subset F_2$ implies $\kappa(F_1) \leq \kappa(F_2)$.

(ii) $\kappa(F_1 \cup F_2) \leq \kappa(F_1) + \kappa(F_2)$.

(iii) If card $E = n$, then $\kappa(E) \leq (n+1)\kappa(1/(n+1))$; the estimate is sharp and attained for $x_1 - x_0 = x_2 - x_1 = \cdots = x_{n+1} - x_n$.

2. Examples of κ-entropy.

(a) $\kappa(s) = s$. We have in this case $\kappa(F) = 1\,(F \neq \varnothing)$, $\kappa(\varnothing) = 0$.

(b) $\kappa(s) = 1\,(0 < s \leq 1)$. Here we have

$$\kappa(F) = \text{card}(F \cup \{a,b\}) - 1 \qquad (F \neq \varnothing).$$

(c) $\kappa(s) = s(1 - \log s)$. The corresponding entropy will be denoted by $\kappa_s(F)$ and called the *Shannon entropy* of F (relative to $[a,b]$).

(d) $\kappa(s) = s^\alpha$. Here $\kappa(F) = \kappa_{l,\alpha}(F)$ is the *Lipschitz entropy* $(0 < \alpha < 1)$.

(e) $\kappa(s) = (1 - \frac{1}{2}\log s)^{-1}$; $\kappa(F) = \kappa_d(F)$ is the *Dini entropy.*

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