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EXPLICIT RELAXATION OF A VARIATIONAL PROBLEM IN OPTIMAL DESIGN[1]

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Our goal is to construct a quasiconvex function Φ such that

$$(1) \qquad \inf_{u|_{\partial\Omega}=F} \int_\Omega (1_{\operatorname{supp} \nabla u} + |\nabla u|^2)\, dx = \inf_{u|_{\partial\Omega}=F} \int_\Omega \Phi(\nabla u)\, dx$$

for vector-valued functions u on Lipschitz domains $\Omega \subset \mathbf{R}^2$. The right side of (1) is the relaxation of the left, cf. [**1**]. Each infimum is over $u \in H^1(\Omega; \mathbf{R}^N)$, $1_{\operatorname{supp} \nabla u}$ denotes the characteristic function of the support of ∇u, and $|\nabla u|^2 = \sum(\partial u^i/\partial x_j)^2$.

The left side of (1) is a problem of optimal design: it minimizes $\operatorname{Area}(\Omega \setminus S) + \int_\Omega |\nabla u_S|^2\, dx$, among all sets $S \subset \Omega$, where u_S solves the variational problem

$$(2) \qquad \inf \left\{ \int_\Omega |\nabla u|^2\, dx \colon u|_{\partial\Omega} = F,\ \nabla u = 0 \text{ on } S \right\}.$$

An application will be described below.

For some choices of Ω and F, this optimal design problem has no solution; in other words, the infimum on the left side of (1) may not be attained. The nonexistence of solutions to related problems has been noted by several authors; see [**4**] and the references given there. Here, it arises because the function

$$(3) \qquad G(\nabla u) = \begin{cases} 1 + |\nabla u|^2, & \nabla u \neq 0, \\ 0, & \nabla u = 0, \end{cases}$$

is not quasiconvex, so the left side of (1) is not lower semicontinuous under weak H^1 convergence. A minimizing sequence $\{u_n\}$ may be highly oscillatory, and $S_n = \{\nabla u_n = 0\}$ may develop increasingly complicated microstructure.

The relaxed problem, on the right, is lower semicontinuous, hence the infimum is attained. In fact, its solutions are precisely the weak limits of minimizing sequences for the left side. The introduction of such a relaxed problem is a standard way of dealing with nonexistence. The method has its roots in the work of L. C. Young and E. J. McShane, and contributions have been made by Morrey, Ball, Ekeland, Temam, and Dacorogna, among others; see [**2**] for further discussion and references.

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