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# THE UNCERTAINTY PRINCIPLE

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ABSTRACT. If a function $\psi(x)$ is mostly concentrated in a box $Q$, while its Fourier transform $\hat{\psi}(\xi)$ is concentrated mostly in $Q'$, then we say $\psi$ is microlocalized in $Q \times Q'$ in $(x, \xi)$-space. The uncertainty principle says that $Q \times Q'$ must have volume at least 1. We will explain what it means for $\psi$ to be microlocalized to more complicated regions $\mathcal{B}$ of volume $\sim 1$ in $(x, \xi)$-space. To a differential operator $P(x, D)$ is associated a covering of $(x, \xi)$-space by regions $\{\mathcal{B}_\alpha\}$ of bounded volume, and a decomposition of $L^2$-functions $u$ as a sum of "components" $u_\alpha$ microlocalized to $\mathcal{B}_\alpha$. This decomposition $u \to (u_\alpha)$ diagonalizes $P(x, D)$ modulo small errors, and so can be used to study variable-coefficient differential operators, as the Fourier transform is used for constant-coefficient equations. We apply these ideas to existence and smoothness of solutions of PDE, construction of explicit fundamental solutions, and eigenvalues of Schrödinger operators. The theorems are joint work with D. H. Phong.

## CHAPTER I: THE *SAK* PRINCIPLE

The uncertainty principle says that a function $\psi$, mostly concentrated in $|x - x_0| < \delta_x$, cannot also have its Fourier transform $\hat{\psi}$ mostly concentrated in $|\xi - \xi_0| < \delta_\xi$, unless $\delta_x \cdot \delta_\xi \geq 1$. This simple fact has far-reaching consequences for PDE, but until recently it was used only in a very crude form. The most significant classical application concerned the eigenvalues of a selfadjoint differential operator

$$A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) \left( \frac{1}{i} \frac{\partial}{\partial x} \right)^\alpha$$

with symbol $A(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$. According to the uncertainty principle, each box

$$\mathcal{B} = \{(x, \xi) |\, |x - x_0| < \delta,\ |\xi - \xi_0| < \delta^{-1}\}$$

should count for one eigenvalue, so the number of eigenvalues of $A(x, D)$ which are less than $K$ should be given approximately as the volume of the set $S(A, K) = \{(x, \xi) \mid A(x, \xi) < K\}$. If $A$ is elliptic and $K \to \infty$, then this "volume-counting" is asymptotically correct (see Weyl [**41**], Carleman [**5**], Hörmander [**23**]). However, volume-counting can also produce grossly inaccurate estimates for systems as simple as two uncoupled harmonic oscillators.

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