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COMPLEMENTING MAPS, CONTINUATION AND GLOBAL BIFURCATION

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ABSTRACT. We state, and indicate some of the consequences of, a theorem whose sole assumption is the nonvanishing of the Leray-Schauder degree of a compact vector field, and whose conclusions yield multidimensional existence, continuation and bifurcation results.

Complementing maps and the Theorem. Let X be a Banach space, m be a positive integer, and $O \subseteq \mathbf{R}^m \times X$ be open. Suppose $f\colon O \to X$ is an m-parameter compact vector field: i.e. $f(\lambda, x) = x - F(\lambda, x)$, for $(\lambda, x) \in O$, where F is continuous and maps bounded sets into relatively compact sets. A continuous map $g\colon O \to \mathbf{R}^m$, which maps bounded sets into bounded sets, will be called a *complement* for $f\colon O \to X$ provided that the Leray-Schauder degree, $\deg((g,f), O, 0)$, is defined and nonzero: $(g,f)((\lambda,x)) \equiv (g(\lambda,x), f(\lambda,x))$, for $(\lambda,x) \in O$, and since O is not assumed to be bounded, "defined" means $(g,f)^{-1}(0)$ is compact.

By cohomology we will mean Čech cohomology with integral coefficients. By dimension of a topological space we mean the Čech-Lebesgue covering dimension, and if $p \in A$, the space A will be said to have dimension at least m at p provided that each neighborhood, in A, of p has dimension at least m.

THEOREM. *Let X be a Banach space, m be a positive integer, and $O \subseteq \mathbf{R}^m \times X$ be open. Suppose that $f\colon O \to X$ is complemented by $g\colon O \to \mathbf{R}^m$. Then there exists a closed connected subset, C, of $f^{-1}(0)$, whose dimension at each point is at least m, and $(*)$ whenever K is a compact subset of C, with $g^{-1}(0) \cap C \subseteq K$, the map of pairs $g\colon (C, C-K) \to (\mathbf{R}^m, \mathbf{R}^m - 0)$ induces a nontrivial map in the mth cohomology group. In particular, $C \cap g^{-1}(0) \neq \varnothing$ and either C is unbounded or $\overline{C} \cap \partial O \neq \varnothing$. In the case when f and g are defined on $\overline{O}$ with $f^{-1}(0) \cap g^{-1}(0) \cap \partial O = \varnothing$, C also has the following properties: if C is bounded, then $\dim(\overline{C} \cap \partial O) \geq m-1$, when $m > 1$, and $\overline{C} \cap \partial O$ has at least two points, when $m = 1$; if $g\colon f^{-1}(0) \cap \overline{O} \to \mathbf{R}^m$ is proper and $\dim(\overline{C} \cap \partial O) < m-1$, then $g(\overline{C}) = \mathbf{R}^m$.*

SKELETON OF THE PROOF. Since $\deg((g,f), O, 0) \neq 0$, by using the cup-product in cohomology, it follows that whenever K is compact and $g^{-1}(0) \subseteq K \subseteq f^{-1}(0)$ the map $g\colon (f^{-1}(0), f^{-1} - K) \to (\mathbf{R}^m, \mathbf{R}^m - 0)$ is cohomologically nontrivial. Passing to the limit over all such K's we obtain a nontrivial class, ξ, in the mth Čech cohomology group with compact supports of $f^{-1}(0)$. The continuity of Čech theory enables us to choose a set, C, which is minimal

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