BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

0273-0979/82/0000-0982/$01.75

*Functional integration and quantum physics*, by Barry Simon, Academic Press, New York, 1979, ix + 296 pp., $29.50.

*Quantum physics, a functional integral point of view*, by James Glimm and Arthur Jaffe, Springer-Verlag, New York, 1981, xx + 417 pp., $16.80.

These two books have strikingly similar titles, but the topics discussed are almost disjoint. Common to both is the approach to problems of quantum physics by first studying them in imaginary time.

If we know all about the selfadjoint operator $H$, then we know all about the solution $\psi = e^{-itH}\psi_0$ of the Schrödinger equation; but if we know all about $f = e^{-tH}f_0$ (the solution of the Schrödinger equation in "imaginary time"), then we know all about $H$.

The simplest $H$ of interest in quantum physics is of the form $-\frac{1}{2}\Delta + V$ on $L^2(\mathbf{R}^\nu)$, where $V$ is the operator of multiplcation by a function. This case is discussed in depth by Simon. The Feynman-Kac formula gives an explicit expression for the kernel of the integral operator $e^{-tH}$.

$$(1) \qquad e^{-tH}(a, b) = \int \exp\left(-\int_0^t V(\omega(s))\, ds\right) d\mu_{0,a,b;t}.$$

The integration is over the space of all paths $\omega$: $\mathbf{R} \to \mathbf{R}^\nu$ and $\mu_{0,a,b;t}$ is the condition Wiener measure for paths starting at $a$ at time 0 and ending at $b$ at time $t$.

As a simple example of the power of functional integration, Simon gives Symanzik's proof of the Golden-Thompson inequality

$$(2) \qquad \operatorname{Tr} e^{-tH} \leqslant \int \frac{d^\nu p\, d^\nu x}{(2\pi)^\nu} e^{-t(p^2/2 + V(x))}.$$

The trace on the left contains a wealth of information on the distribution of eigenvalues; the integral on the right is a classical phase space integral. To evaluate the trace, set $a = b$ in (1) and integrate over $\mathbf{R}^\nu$. We can estimate

$$\exp\left(-\int_0^t V(\omega(s))\, ds\right) \leqslant \frac{1}{t}\int_0^t \exp(-tV(\omega(s)))\, ds,$$

by Jensen's inequality, and then (2) follows. Details are in Simon's Theorem 9.2; the point is that the usual tools of measure theory (monotone convergence, Jensen's inequality, etc.) can be brought to bear when working in imaginary time since we have a probability measure on paths. Simon discusses a vast number of other applications of functional integration to the Schrödinger equation. Of particular interest to probabilists, because of the occurrence of stochastic integrals, is the discussion of Schrödinger operators with magnetic fields.