BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

HOOK YOUNG DIAGRAMS, COMBINATORICS AND REPRESENTATIONS OF LIE SUPERALGEBRAS

A. BERELE[1] AND A. REGEV[2]

Let V be a finite-dimensional F-vector space, $\operatorname{char}(F) = 0$. Schur introduced the action of the symmetric group S_n on $V^{\otimes n}$ and was then able to determine the representation theory of the general linear group $\mathrm{GL}(V)$ [**9**]. His work was later completed by H. Weyl [**10**]. This work connects that representation theory with combinatorics via standard and semistandard Young tableaux and via the Schur functions (cf. [**6**]). Many of the objects in this theory are parametrized by the Young diagrams in a strip.

In this work we introduce a slightly more general permutation action of S_n on $V^{\otimes n}$ and then describe how most of the above theory generalizes. The main feature here is that most of the generalized objects are parametrized by the partitions inside a hook.

The action. Let $k, l \geq 0$, $k+l > 0$, T and U disjoint vector spaces, $\dim T = k$, $\dim U = l$, and $V = T \oplus U$. We define a new right action of S_n on $V^{\otimes n}$, i.e., a map $\psi\colon S_n \to \operatorname{End}_F(V^{\otimes n})$, based on Schur's original action and on the functions $f_I\colon S_n \to \{\pm 1\}$ [**5**] as follows. Choose bases $t_1, \ldots, t_k \in T$, $u_1, \ldots, u_l \in U$. These induce a basis of $V^{\otimes n}$. Let $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$, $v_1, \ldots, v_n \in \{t_1, \ldots, u_l\}$ be such a basis element, let $I = \{i | v_i \in U\}$ and let $\sigma \in S_n$. Then

$$(v_1 \otimes \cdots \otimes v_n)\psi(\sigma) \underset{\mathrm{DEF}}{=} f_I(\sigma)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}).$$

Extend $\psi(\sigma)$ to all of $V^{\otimes n}$ by linearity: $\psi(\sigma) \in \operatorname{End}(V^{\otimes n})$. As usual, we now extend ψ to FS_n, then check that $\psi\colon FS_n \to \operatorname{End}(V^{\otimes n})$ is an (associative) algebra homomorphism.

The question. It is well known that $FS_n = \sum_{\lambda \in \operatorname{Par}(n)} \oplus I_\lambda$, where $\operatorname{Par}(n)$ denotes the set of partitions of n and where each I_λ is a simple algebra. It follows that for some $\Gamma = \Gamma(k, l; n) \subseteq \operatorname{Par}(n)$, $\psi(FS_n) \cong \sum_{\lambda \in \Gamma} \oplus I_\lambda$, and the basic question here is to describe Γ. Letting $B(k, l; n)$ be the centralizer of $\psi(FS_n)$ in $\operatorname{End}_F(V^{\otimes n})$, the decomposition of $V^{\otimes n}$ into irreducible $\psi(FS_n)$ or $B(k, l; n)$ modules, will be given by the classical theory of Schur.

The answer, which extends a theorem of Weyl is

THE HOOK THEOREM. *Let*

$$H(k, l; n) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \in \operatorname{Par}(n) | \lambda_j \leq l \text{ if } j \geq k+1\}$$

Received by the editors April 26, 1982.

1980 *Mathematics Subject Classification*. Primary 05A15, 05A17, 17A70, 17B10, 20C30; Secondary 16A38, 81C40.

[1]Supported by a grant from the Bantrell Foundation.

[2]Partially supported by grant M(T) from the Israel Academy of Sciences & Humanities.