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Numerical analysis of variational inequalities, by R. Glowinski, J. L. Lions and R. Trémolières, Studies in Mathematics and its Applications, vol. 8, North-Holland, Amsterdam, 1981, xxx + 778 pp., $109.75.

Consider the problem of minimizing a real-valued function f over a space V. If u attains the minimum and f is differentiable at u, then $f'[u] = 0$. On the other hand, if K is a convex subset of V and u is optimal for the problem

$$\text{minimize } \{f(v): v \in K\}, \tag{1}$$

then an inequality holds,

$$f'[u](v-u) \geqslant 0 \qquad \forall v \in K. \tag{2}$$

Loosely speaking, (2) says that f increases when we move from u into K. The book by Glowinski, Lions, and Trémolières studies numerical aspects of (1) and (2) for a broad class of physical problems.

The obstacle problem illustrates the type of inequality included in their analysis: Given an open set $\Omega \subset R^2$ and functions $f \in \mathcal{L}^2(\Omega)$ and $\psi \in \mathcal{H}^2(\Omega)$,

$$\text{minimize } \int_\Omega \left\{ \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2fv \right\} dx\, dy$$

subject to $v \in \mathcal{H}^1(\Omega)$, $v = 0$ on $\partial\Omega$, $v \geqslant \psi$ almost everywhere in Ω. Here $\mathcal{L}^2(\Omega)$ is the space of real-valued functions that are square integrable on Ω, and $\mathcal{H}^k(\Omega) \subset \mathcal{L}^2(\Omega)$ is the Sobolev subspace consisting of functions whose derivatives through order k lie in $\mathcal{L}^2(\Omega)$. The function ψ is the obstacle. In this context, it can be shown [3] that the inequality (2) is equivalent to the relations

$$\left.\begin{array}{l} u \geqslant \psi \\ f \geqslant \Delta u \\ (f - \Delta u)(\psi - u) = 0 \end{array}\right\} \quad \text{almost everywhere in } \Omega.$$

These relations tell us that $u = \psi$ on part of Ω while $u > \psi$ and $\Delta u = f$ on the complement. The curve that forms the boundary of $\{x \in R^2: u(x) > \psi(x)\}$ is often called the *contact set*.

Many physical problems have the form (1) or (2), and the book by Duvaut and Lions [6] is a good reference on this subject. For example, in plasticity theory, the stress is constrained to lie inside a yield surface. The stress potential for an elastic-perfectly plastic cylindrical bar undergoing torsion is the solution to the problem

$$\text{minimize } \int_\Omega \left\{ \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2fv \right\} dx\, dy$$

subject to $v \in \mathcal{H}^1(\Omega)$, $v = 0$ on $\partial\Omega$, $(\partial v/\partial x)^2 + (\partial v/\partial y)^2 \leqslant 1$ almost everywhere in Ω where Ω is the bar's cross-section, and the constraint $|\nabla v|^2 \leqslant 1$ is