BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 1, January 1983

SIMULTANEOUS SIMILARITY OF MATRICES

SHMUEL FRIEDLAND

Let M_n be the set of $n\times n$ matrices over the algebraically closed field k, G_n the general linear group in M_n, $M_{n,m} = M_n \times \cdots \times M_n (m+1$ times). G_n acts naturally on $M_{n,m}$ by the conjugation $TM_{n,m}T^{-1}$. For $\alpha = (A_0, \ldots, A_m) \in M_{n,m}$ denote by $\operatorname{orb}(\alpha)$ the orbit of α in $M_{n,m}$,

$$\operatorname{orb}(\alpha) = \{\beta \in M_{n,m}, \beta = T\alpha T^{-1} = (TA_0T^{-1}, \ldots, TA_mT^{-1}), T \in GL_n\}.$$

It is a well-known problem to classify $\operatorname{orb}(\alpha)$ for $m \geq 1$. See for example [**2**]. Rosenlicht in [**3**] outlined a general classification based on the ideas of algebraic geometry. The classification consists of a finite number of steps. In each step we get an algebraic irreducible variety V in $M_{n,m}$ which is invariant, that is $TVT^{-1} = V$ for all $T \in G_n$. Then, we consider $k(V)^G$—the field of rational functions on V which are invariant, i.e. these functions are constant on $\operatorname{orb}(\alpha)$. It follows that $k(V)^G$ is finitely generated, let us say by $\chi_1, \ldots, \chi_j$. Then there exists locally closed algebraic invariant set V^0 in V such that for any $\alpha \in V^0 \chi_1, \ldots, \chi_j$ are well defined on $\operatorname{orb}(\alpha)$ and the values of χ_k, $k = 1, \ldots, j$, on $\operatorname{orb}(\alpha)$ determine this orbit uniquely in V^0.

The purpose of this announcement is to describe explicitly the open invariant varieties V^0 together with the invariant rational functions $\varphi_1, \ldots, \varphi_k$ defined on V^0 such that the values of $\varphi_1, \ldots, \varphi_k$ on $\operatorname{orb}(\alpha)$ determine a finite number of orbits. We also describe some results on orbits in $S_{n,m} = S_n \times \cdots \times S_n$ $(m+1$ times) ($S_n =$ the set of $n \times n$ complex symmetric matrices) under the action of O_n-complex orthogonal group in M_n.

For $\alpha = (A_0, \ldots, A_m)$, $\beta = (B_0, \ldots, B_m)$ let $\operatorname{adj}(\alpha, \beta)$: $M_n \to M_{n,m}$ be a linear operator given by $\operatorname{adj}(\alpha,\beta)(X) = (A_0X - XB_0, \ldots, A_mX - XB_m)$.

We identify $\operatorname{adj}(\alpha, \alpha)$ with $\operatorname{adj}(\alpha)$. Let $r(\alpha, \beta)$ and $r(\alpha)$ be the ranks of $\operatorname{adj}(\alpha, \beta)$ and $\operatorname{adj}(\alpha)$ respectively. Then $r(\alpha)$ is the first discrete invariant of $\operatorname{orb}(\alpha)$ and it gives the dimension of the manifold $\operatorname{orb}(\alpha)$. Suppose that $\beta \in \operatorname{orb}(\alpha)$. Then one easily shows that $r(\alpha, \beta) = r(\alpha)$. Fix α and consider all $\xi \in M_{n,m}$ which satisfy the inequality

$$\mathcal{X}(\alpha) = \{\xi, r(\alpha, \xi) \leq r,\ \xi = (X_0, \ldots, X_m) \in M_{n,m}\}. \tag{1}$$

The set $\mathcal{X}(\alpha)$ is an algebraic set in $M_{n,m}$ which can be given by

$$N(r) = \binom{n^2}{r+1}\binom{n^2 \quad (m+1)}{r+1} \text{polynomial equations.}$$

Received by the editors December 22, 1981.

1980 *Mathematics Subject Classification*. Primary 14D25, 14L30, 15A21.

Key words and phrases. Simultaneous similarity, invariant functions, symmetric matrices.

0273-0979/82/0000-1031/$01.50