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ARITHMETIC CHARACTERIZATIONS OF SIDON SETS

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ABSTRACT. Let $\hat{G}$ be any discrete Abelian group. We give several arithmetic characterizations of Sidon sets in $\hat{G}$. In particular, we show that a set Λ is a Sidon set iff there is a number $\delta > 0$ such that any finite subset A of Λ contains a subset $B \subset A$ with $|B| \geq \delta |A|$ which is quasi-independent, i.e. such that the only relation of the form $\sum_{\lambda \in B} \epsilon_\lambda \lambda = 0$, with ϵ_λ equal to ± 1 or 0, is the trivial one.

Let G be a compact Abelian group and let $\hat{G}$ be the dual group. For any f in $L_2(G)$, we denote by $\hat{f}$ the Fourier transform of f. A subset Λ of $\hat{G}$ is called a Sidon set if there is a constant K with the following property: all the trigonometric polynomials f, such that $\hat{f}$ is supported by Λ, satisfy

$$\sum |\hat{f}(\gamma)| \leq K \|f\|_{C(G)}.$$

We will denote by $S(\Lambda)$ the smallest constant K with this property. In the theory of Sidon sets (cf. e.g. [**2**]), there has always been considerable interest in the relations between this analytical definition and the arithmetic properties of the set Λ (in particular, in the case $G = \mathbf{T}$ and $\Lambda \subset \mathbf{Z}$). The aim of this note is to announce several arithmetic characterizations of Sidon sets.

Let us make more precise what we mean here by "arithmetic". We will denote by R_Λ the set of relations (with coefficients in $\{-1,0,1\}$) satisfied by Λ, i.e. the set of all finitely supported families $(\epsilon_\lambda)_{\lambda\in\Lambda}$ in $\{-1,0,1\}^\Lambda$ such that $\sum_{\lambda\in\Lambda} \epsilon_\lambda \lambda = 0$.

By an "arithmetic" characterization is usually meant one which depends only on the set R_Λ. In [**1**], Drury[1] proved that such a characterization exists, but he could not produce any explicit one. Precisely, he proved the following: let Λ and Λ' be two sets for which there is a bijection $\phi\colon \Lambda' \to \Lambda$ such that the map $\tilde{\phi}\colon R_\Lambda \to R_{\Lambda'}$, defined by $\tilde{\phi}((\epsilon_\lambda)_{\lambda\in\Lambda}) = (\epsilon_{\phi(\lambda')})_{\lambda'\in\Lambda'}$, is also a bijection. Then, Λ is a Sidon set iff the same is true for Λ'. In other words, the property of "being a Sidon set" is determined by R_Λ. We give below several *explicit* arithmetic characterizations, from which the preceding result of Drury follows as a corollary.

To state our results, we will need some notation and terminology. We will denote by I_Λ the set of all finitely supported families $(\epsilon_\lambda)_{\lambda\in\Lambda}$ in $\{-1,0,1\}^\Lambda$. For any γ in $\hat{G}$, we will denote by $R(\gamma,\Lambda)$ the number of ways to write γ as

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[1]Drury considers only relations such that moreover $\sum_{\lambda\in\Lambda} \epsilon_\lambda = 0$, but this difference is not significant, since we can replace Λ by the set $\tilde{\Lambda} \subset \hat{G} \times \mathbf{Z}$ defined by $\tilde{\Lambda} = \{(\lambda, 1) | \lambda \in \Lambda\}$.

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