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DIAGONALIZING MATRICES OVER OPERATOR ALGEBRAS

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1. Introduction. Let A_0 be a C^*-algebra and A be the algebra of $n \times n$ matrices with entries in A_0. If A_0 acting on a (complex) Hilbert space H_0 is a faithful representation of A_0, then A acting as matrices on the n-fold direct sum H of H_0 with itself is a faithful representation of A. As a subalgebra of $B(H)$, the algebra of all bounded operators on H, A acquires an adjoint and norm structure relative to which it is a C^*-algebra. This structure can be described independently of the representations—in particular, the operator in $B(H)$ adjoint to (a_{jk}) is the element of A whose matrix has a_{kj}^* as its j, k entry. If A_0 is the (algebra of) complex numbers C, then A is the algebra of $n \times n$ complex matrices and each normal element a can be "diagonalized"—that is, there is a unitary element u in A such that uau^{-1} has all its nonzero entries on the diagonal.

With A_0 a general C^*-algebra, can each normal element of A be diagonalized? In §2, we give a construction (based on homotopy groups of spheres) to show that this (general) question has a negative answer. The main result is discussed in §3. If A_0 is a von Neumann algebra, diagonalization of normal operators is always possible. More generally,

THEOREM. *If R_0 is a von Neumann algebra, R is the algebra of $n \times n$ matrices over R_0, and S is a commutative subset of R with the property that a^* is in S if a is in S, then there is a unitary element u in R such that uau^{-1} has all its nonzero entries on the diagonal for each a in S.*

2. An example. Let A_0 be the algebra $C(S^4)$ of continuous complex-valued functions on the 4-sphere S^4 and let A be the algebra of 2×2 matrices with entries in A_0. View S^3 as the unit sphere in two-dimensional Hilbert space C^2 and consider the standard action of $SU(2)$ (the group of 2×2 unitary matrices of determinant 1) on C^2. The mapping that takes u in $SU(2)$ to the vector $u(1,0)$ is a homeomorphism of $SU(2)$ onto S^3. From **[2]**, $\pi_4(S^3)$ is the additive group of integers modulo 2. Let u_0 be an essential mapping of S^4 into $SU(2)$ (that is, into S^3). The algebra A can be viewed as continuous mappings of S^4 into $B(C^2)$. Thus u_0 is a unitary (hence normal) element of A. Suppose u is a unitary element of A that diagonalizes u_0. Then $u(p)u_0(p)u(p)^{-1}$ is a 2×2 diagonal matrix over C for each p in S^4. Let $\theta(p)$ be the complex conjugate of the determinant of $u(p)$, let $u_1(p)$ be $\left[\begin{smallmatrix} \theta(p) & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and let $v(p)$ be $u_1(p)u(p)$.

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