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SELF-DUAL CONNECTIONS AND THE TOPOLOGY OF SMOOTH 4-MANIFOLDS

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1. Introduction, statement of result. To any compact oriented 4-manifold X there is associated a quadratic form Q, defined on the cohomology group $H^2(X;\mathbf{Z})$ by $Q(\alpha) = (\alpha \cup \alpha)[X]$. Poincaré duality requires that it be a "unimodular" form—given by a symmetric matrix of determinant ± 1 with respect to any base for the torsion free part of H^2. It is known from arithmetic that there are many such forms that are positive definite and not equivalent (over the integers) to the standard form [**4**, Chapter 5]. The problem of finding which forms are realised by simply-connected 4-manifolds was raised, for example, in [**3**]; a partial answer for smooth 4-manifolds is announced here in the form of

THEOREM. *If X is a smooth, compact, simply-connected oriented 4-manifold with the property that the associated form Q is positive definite, then Q is equivalent, over the integers, to the standard diagonal form.*

As a particular application, the theorem shows that it is impossible to remove smoothly, by surgery, all three hyperbolic factors in a K3 surface (which has quadratic form $E_8 + E_8 + \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$) since this would give a simply-connected smooth 4-manifold with definite form $E_8 + E_8$.

2. Method of proof. I give, in this note, an outline of the proof; a detailed account will appear soon. The idea of the proof is to exploit topological information that emerges from a study of "self-dual connections" or "instantons"; I take [**1**] as a general reference for background in this area, and for notation. Suppose throughout that X is a 4-manifold satisfying the hypotheses of the theorem, and that we are given some Riemannian metric.

There is, up to isomorphism, a unique principal $SU(2)$ bundle P over X with characteristic class $c_2(P)[X] = -1$. One forms the space of all equivalence classes of connections on P as the quotient of the affine space $\mathcal{A}$ of connections by the action of the "gauge group" $\mathcal{G}$ of automorphisms of P. A Hausdorff topology descends to $\mathcal{A}/\mathcal{G}$ and the dense open subset representing *irreducible* connections can be made into a Banach manifold. On the other hand a *reducible* connection corresponds to a reduction of P to an S^1 bundle and in the neighbourhood of such a point the space $\mathcal{A}/\mathcal{G}$ has the structure of

$$\text{(Real Banach Space)} \times \text{(Complex Banach Space}/S^1).$$

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