BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 3, 1982

REAL RELATIVE COHOMOLOGY OF FINITE-CODIMENSION GERMS

BY CLAUDE ANDRÉ ROCHE

1. Relative and vanishing cohomology. Let P be a germ at $0 \in \mathbf{R}^n$ of a C^∞ function of finite codimension (see [7]). Let $\mathcal{E}_n$ be the ring of germs at 0 of smooth functions: $\mathbf{R}^n \to \mathbf{R}$, m_n^∞ the ideal of flat germs of $\mathcal{E}_n$ (i.e. with null Taylor expansion) and $\mathcal{F}_n$ the ring of real formal series in n indeterminates $X_1, \ldots, X_n$. Set $\Lambda^\cdot$ (respectively $\Lambda^\cdot_\infty$, $\hat{\Lambda}^\cdot$) to be the de Rham complex of germs of smooth forms (resp. of forms of $m_n^\infty \Lambda^\cdot$, of "forms" in the module over $\mathcal{F}_n$ spanned by $dX_1, \ldots, dX_n$). We construct a relative de Rham complex $(\Lambda^\cdot_{\text{rel}}, d_{\text{rel}})$ (resp. $(\Lambda^\cdot_{\infty\,\text{rel}}, d_{\text{rel}})$, $(\hat{\Lambda}^\cdot_{\text{rel}}, d_{\text{rel}})$) using Leibniz rule, $\Lambda^\cdot_{\text{rel}} = \Lambda^\cdot / dP \wedge \Lambda^\cdot$ (resp. $\Lambda^\cdot_{\infty\,\text{rel}} = \Lambda^\cdot_\infty / dP \wedge \Lambda^\cdot_\infty$, $\hat{\Lambda}^\cdot_{\text{rel}} = \hat{\Lambda}^\cdot / dj^\infty P \wedge \hat{\Lambda}^\cdot$, j^∞ the jet epimorphism, $\mathcal{E}_n \to \mathcal{F}_n$) this module is endowed with a natural structure of $\mathcal{E}_1$ (resp. m_1^∞, $\mathcal{F}_1$)-module through P. By the chain rule d_{rel} is linear. Define $H^\cdot_{\text{rel}}$ (resp. $H^\cdot_{\infty\,\text{rel}}$, $\hat{H}^\cdot_{\text{rel}}$) as the cohomology module of the relative de Rham complex of smooth (resp. flat, formal) forms.

Relative cohomology appears in the study of hypersurface singularities. In the complex case, E. Brieskorn [2] proved that holomorphic relative cohomology is determined by the formal one (Bloom's theorem). In fact this cohomology is completely determined by the vanishing cohomology sheaf in the case of an isolated singularity (Brieskorn-Sebastiani).

We can define the real vanishing cohomology of P as the stalk at 0 of the sheaf $R^\cdot P_*(\underline{\mathbf{R}})$, where $\underline{\mathbf{R}}$ is the constant sheaf on a small ball B at the origin of $\mathbf{R}^n$. By the fibration theorem, P if restricted to $B \cap P^{-1}((-\eta, \eta) - \{0\})$ with η small, is a fibre projection and $R^\cdot P_*(\underline{\mathbf{R}})$ a real local system. Hence $R^i P_*(\underline{\mathbf{R}})$ is trivial over $(-\eta, 0)$ and $(0, \eta)$, the stalk being isomorphic to $\mathbf{R}^{a_{-i}}$ and $\mathbf{R}^{a_i}$ respectively. Let b_i be $a_{-i} + a_i$ then b_i is the dimension of the ith vanishing cohomology space.

THEOREM 1. *The m_1^∞-module $H^k_{\infty\,\text{rel}}$ is free of rank b_k.*

THEOREM 2. (a) *The sequence $0 \to \Lambda^\cdot_{\infty\,\text{rel}} \to \Lambda^\cdot_{\text{rel}} \to \hat{\Lambda}^\cdot_{\text{rel}} \to 0$ is exact over $0 \to m_n^\infty \to \mathcal{E}_n \to \mathcal{F}_n \to 0$.*

(b) *The long exact homology sequence associated to it has null connecting morphism.*

The first part is easy for finite-codimension germs and false for infinite-codimension ones. The second uses the result of Bloom-Brieskorn-Sebastiani except in degree 0. In this case a result of Moussu on Left equivalence of germs is used.

Received by the editors May 26, 1982.

1980 *Mathematics Subject Classification*. Primary 58C27, 32B30; Secondary 32C05.

0273-0979/82/0000-0799/$01.75