BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 1, July 1982

BROWNIAN MOTION, GEOMETRY, AND GENERALIZATIONS OF PICARD'S LITTLE THEOREM

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ABSTRACT. Brownian motion is introduced as a tool in Riemannian geometry, and it is shown to be useful in the function theory of manifolds, as well as in the study of maps between manifolds. As applications, a generalization of Picard's little theorem, and a version of it for Riemann surfaces of large genus are given.

1. Picard's theorem for nonhyperbolic manifolds. Let M and N be complete Riemannian manifolds with metrics ${}^Mg$, ${}^Ng$, resp. Assume $F\colon M \longrightarrow N$ is a C^2 map. F is said to be *harmonic* [2] if its second fundamental form has trace 0. Define the tensor

$$\xi^{\alpha\beta}(x) = {}^Mg_{ij}\left[\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}\right](x), \; x \in M.$$

Since $(\xi^{\alpha\beta}(x))$ is a symmetric matrix, its eigenvalues are nonnegative, and we may order them as follows: $\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_n(x) \geqslant 0$. F is said to be *K-quasiconformal* [5] if $\lambda_1(x) \leqslant K^2\lambda_n(x)$ for all $x \in M$.

We define polar coordinates (r, θ) on N via the exponential map. There will be two restrictions on the curvature of N:

(i) *The sectional curvatures of N are bounded below by $-L^2 < 0$.*

(ii) *Each of the sectional curvatures at $(r, \theta) \in N$ determined by dr and some other tangent vector, is bounded above by $K(r)$, where $K(r)$ satisfies* (a) *for some $\epsilon > 0$, $-K(r) \sim r^{2\epsilon-2}$*; (b) *there exists a C^∞ solution $u(r)$ of the equation*

$$u''(r) = K(r)u(r), \qquad u(0) = 0, \quad u'(0) = 1,$$

and $u'(r)$ is always positive.

(Note that such a solution can always be found if $K(r)$ is negative.)

THEOREM 1. *Suppose M and N are as above with the curvature of N satisfying* (i) *and* (ii). *Then, if Brownian motion on M has trivial tail σ-field, every K-quasiconformal harmonic map $F\colon M \longrightarrow N$ is constant.*

Received by the editors January 26, 1982 and, in revised form, March 23, 1982.
1980 *Mathematics Subject Classification*. Primary 32H25, 53C21, 60J65.

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