BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 1, July 1982

THE WHITEHEAD CONJECTURE AND SPLITTING $B(\mathbf{Z}/2)^k$

BY N. J. KUHN, S. A. MITCHELL AND S. B. PRIDDY[1]

1. Introduction. In this note we present a circle of ideas with which the first author has proved G. Whitehead's conjecture concerning symmetric products of the sphere spectrum, i.e.

$$i_*: \pi_* SP^{2^k} S^0 \longrightarrow \pi_* SP^{2^{k+1}} S^0$$

is zero on the 2-components in positive dimensions [**Mi**, Conjecture 84]. Equivalently, the natural sequence of spectra

$$\cdots \longrightarrow L(3) \xrightarrow{\delta_2} L(2) \xrightarrow{\delta_1} L(1) \xrightarrow{\delta_0} L(0) \longrightarrow H\mathbf{Z},$$

localized at 2, is exact on homotopy groups. Here $H\mathbf{Z}$ is the integral Eilenberg-Mac Lane spectrum, $L(0) = S^0$, and $L(k) = \Sigma^{-k} SP^{2^k} S^0 / SP^{2^{k-1}} S^0$. Since $L(1) = \mathbf{R}P^\infty$ [**JTTW**], exactness at $L(0)$ is equivalent to the Kahn-Priddy theorem [**KP**].

In establishing this geometric resolution, it was found necessary to show that $L(k)$ is projective in an appropriate sense. Regarding suspension spectra as free objects, wedge summands of suspension spectra can be considered projective. The second and third authors have shown that $L(k)$ is projective by using the Steinberg idempotent [**S**] for $\mathbf{F}_2 GL_k(\mathbf{F}_2)$ to prove that $L(k)$ is a wedge summand in the suspension spectrum of $B(\mathbf{Z}/2)^k = \mathbf{R}P^\infty \times \cdots \times \mathbf{R}P^\infty$.

It appears likely that our results also hold true for odd primes and tentative results have been obtained in this direction. Throughout this paper all spaces and spectra are localized at 2 and all cohomology groups are taken with $\mathbf{Z}/2$ coefficients unless otherwise specified.

Details will appear elsewhere.

2. Symmetric products. If X is a space the *symmetric product* $SP^k X = X^k/\Sigma_k$ is the set of unordered k-tuples $\langle x_1, \ldots, x_k\rangle$, $x_i \in X$. For pointed X, $\langle x_1, \ldots, x_k\rangle \longrightarrow \langle x_1, \ldots, x_k, *\rangle$ defines an inclusion $SP^k X \xrightarrow{i} SP^{k+1} X$. The limit $SP^\infty X$ satisfies $\pi_* SP^\infty X = \widetilde{H}_*(X; \mathbf{Z})$ by the Dold-Thom theorem [**DT**]. There is also a natural pairing $SP^k X \wedge SP^l Y \xrightarrow{\wedge} SP^{kl}(X \wedge Y)$ defined by
$\langle x_1, \ldots, x_k\rangle \wedge \langle y_1, \ldots, y_l\rangle \longrightarrow \langle x_1 \wedge y_1, \ldots, x_i \wedge y_j, \ldots, x_k \wedge y_l\rangle$.
In particular $S^1 \wedge SP^k Y \xrightarrow{\wedge} SP^k(S^1 \wedge Y)$ and so the symmetric product construction passes to spectra. For the sphere spectrum, $SP^\infty S^0 = H\mathbf{Z}$. A mod 2

Received by the editors January 21, 1982.
1980 *Mathematics Subject Classification*. Primary 55P42, 55R35; Secondary 20C20.
[1]Partially supported by NSF grant MCS-7827592.