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ON DEFINING RELATIONS OF CERTAIN INFINITE-DIMENSIONAL LIE ALGEBRAS[1]

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ABSTRACT. In this note we prove a conjecture stated in [2] about defining relations of the so-called Kac-Moody Lie algebras. In the finite-dimensional case this is Serre's theorem [5]. The basic idea is to map the ideal of relations into a Verma module and then to use the (generalized) Casimir operator (cf. [3, 4]).

1. The main statements. Let $A = (a_{ij})$ be an $n \times n$ matrix over a field $\mathbf{F}$. Denote by $\widetilde{\mathfrak{g}}(A)$ the Lie algebra over $\mathbf{F}$ with $3n$ generators e_i, f_i, h_i, $i \in I = \{1, \ldots, n\}$ and the following defining relations ($i, j \in I$):

$$(1) \qquad [e_i, f_j] - \delta_{ij} h_i, \quad [h_i, h_j], \quad [h_i, e_j] - a_{ij} e_j, \quad [h_i, f_j] + a_{ij} f_j.$$

Set $\Gamma = \mathbf{Z}^n$, $\Gamma_+ = \{(k_1, \ldots, k_n) \in \Gamma | k_i \geqslant 0\} \backslash \{0\}$ and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the standard basis of Γ. Setting $\deg e_i = -\deg f_i = \alpha_i$ for $i \in I$ defines a Γ-gradation $\widetilde{\mathfrak{g}}(A) = \bigoplus_{\alpha \in \Gamma} \widetilde{\mathfrak{g}}_\alpha$. Let $\widetilde{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in \Gamma_+} \widetilde{\mathfrak{g}}_{\pm\alpha}$ and $\mathfrak{h} = \widetilde{\mathfrak{g}}_0$. Then $\widetilde{\mathfrak{n}}_+$ and $\widetilde{\mathfrak{n}}_-$ are free Lie algebras over $\mathbf{F}$ with systems of free generators $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$, respectively, and $\widetilde{\mathfrak{g}}(A) = \widetilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \widetilde{\mathfrak{n}}_+$ (direct sum of vector spaces), so that $\widetilde{\mathfrak{g}}_{\alpha_i} = \mathbf{F}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbf{F}f_i$ for $i \in I$, and $\mathfrak{h} = \bigoplus_i \mathbf{F}h_i$ [2, Chapter I]. Define $(\alpha \mapsto \overline{\alpha}) \in \mathrm{Hom}_{\mathbf{Z}}(\Gamma, \mathfrak{h}^*)$ by $\overline{\alpha}_i(h_j) = a_{ji}$ for $i, j \in I$.

Let $\mathfrak{r}$ be the sum of all graded ideals in $\widetilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially. We have the induced gradation $\mathfrak{r} = \bigoplus_{\alpha \in \Gamma} \mathfrak{r}_\alpha$. Setting $\mathfrak{r}_\pm = \mathfrak{r} \cap \widetilde{\mathfrak{n}}_\pm$, we obtain that $\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-$ is a direct sum of ideals.

Our main result is the following.

THEOREM 1. *For* $\alpha = (k_1, \ldots, k_n) \in \Gamma$ *set*

$$T_\alpha = \sum_{1 \leqslant i < j \leqslant n} a_{ij} k_i k_j + \sum_{1 \leqslant i \leqslant n} a_{ii} \frac{1}{2}(k_i^2 - k_i)$$

and assume that the matrix A is symmetric. Then the ideal $\mathfrak{r}_+$ (*resp.* $\mathfrak{r}_-$) *is generated as an ideal in* $\widetilde{\mathfrak{n}}_+$ (*resp.* $\widetilde{\mathfrak{n}}_-$) *by those* $\mathfrak{r}_\alpha$ (*resp.* $\mathfrak{r}_{-\alpha}$) *for which* $\alpha \in \Gamma_+ \backslash \Pi$ *and* $T_\alpha = 0$.

COROLLARY [4, THEOREM 1]. *If* $T_\alpha \neq 0$ *for all* $\alpha \in \Gamma_+ \backslash \Pi$, *then* $\mathfrak{r} = 0$.

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