BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 5, Number 2, September 1981

GROTHENDIECK-RIEMANN-ROCH FOR COMPLEX MANIFOLDS

BY NIGEL R. O'BRIAN, DOMINGO TOLEDO[1] AND YUE LIN L. TONG[1]

For a coherent analytic sheaf $\mathcal{F}$ on a complex manifold X and a holomorphic map $f: X \longrightarrow Y$ to a complex manifold Y, with f proper on the support of $\mathcal{F}$, we prove a Grothendieck-Riemann-Roch formula as in [3] relating the Todd classes of X, Y and the Chern character of $\mathcal{F}$ and its direct images. This is the first example of a Riemann-Roch theorem for general mappings of complex manifolds which are not necessarily projective varieties. For the history of this problem see [2, 7, 8].

The theorem of Grauert shows that the direct image sheaves $R^if_*\mathcal{F}$ are coherent and we prove the formula

$$\sum_i (-1)^i \operatorname{ch}(R^if_*\mathcal{F})\operatorname{Todd}(Y) = f_*(\operatorname{ch}(\mathcal{F})\operatorname{Todd}(X)).$$

The characteristic classes of X and Y are defined as in [1] and the Chern characters as in [9], so that all classes lie in Hodge cohomology. The theorem thus relates analytic, rather than topological, invariants of $\mathcal{F}$ and its direct images. The same general methods should also lead to the corresponding statements for topological invariants in singular cohomology [2] and perhaps other cohomology theories. The de Rham Chern classes defined in [6] should be relevant in this context.

The proof is based throughout on the techniques of twisting cochains and local formulae introduced in [12] and developed in subsequent papers, and follows Grothendieck's approach to the extent that we factor f as $\pi \circ \Gamma$ where $\Gamma: X \longrightarrow X \times Y$ is the graph of f and $\pi: X \times Y \longrightarrow Y$ is the projection. We first prove the following two special cases.

THEOREM A. *Suppose $\iota: X \longrightarrow Z$ is a closed embedding with the property that there exists a holomorphic retraction $\rho: Z \longrightarrow X$ of maximal rank in a neighbourhood of $\iota(X)$. Then*

$$\operatorname{ch}(\iota_*\mathcal{F}) = \iota_*(\operatorname{Todd}(N)^{-1}\operatorname{ch}(\mathcal{F}))$$

where N is the normal bundle of X in Z.

Received by the editors March 7, 1981.
1980 *Mathematics Subject Classification*. Primary 14C20, 32L10.
([1]) Partially supported by NSF Grants MCS 79-02753 and MCS 79-03798.

0002-9904/81/0000-0406/$01.75