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TWO WEIGHTS WITH ORTHOGONAL SUPPORTS BUT EQUAL ON A DENSE *-SUBALGEBRA

BY A. VAN DAELE

ABSTRACT. We construct two normal semifinite weights φ and ψ on $\mathcal{B}(H)$ with orthogonal supports and such that $\varphi(x) = \psi(x)$ for x in a weakly dense *-subalgebra M_0 contained in $\mathfrak{M}_\varphi$ and $\mathfrak{M}_\psi$. The example is based on the existence of a pair of positive self-adjoint operators h and k on H with orthogonal supports and such that $\|h\xi\| = \|k\xi\|$ for ξ in a dense subspace $\mathcal{D}_0$ contained in $\mathcal{D}(h) \cap \mathcal{D}(k)$.

By a slight modification we obtain two commuting faithful normal semifinite weights on $\mathcal{B}(H)$ that agree on a weakly dense *-subalgebra. This shows that the condition of invariance of this subalgebra for the modular automorphism group may not be omitted in the theorem of Pedersen and Takesaki on equality of weights [2, Proposition 5.9].

Let M be a von Neumann algebra. If φ is a normal semifinite weight on M denote $\mathfrak{N}_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}$ and $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi$. It is well known in the theory of weights that $\mathfrak{N}_\varphi$ is a left ideal and that $\mathfrak{M}_\varphi$ is a *-subalgebra spanned by its positive part $\mathfrak{M}_\varphi^+$ which is equal to $\{x \in M^+ \mid \varphi(x) < \infty\}$. The weight has a unique extension, which we still denote by φ, to a linear functional on $\mathfrak{M}_\varphi$. Because φ is assumed to be semifinite, the subalgebra $\mathfrak{M}_\varphi$ is weakly dense.

We will only be concerned with weights on the von Neumann algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H. In the next proposition we will use the notation $\xi \otimes \eta$ for the rank one operator on H defined by $(\xi \otimes \eta)\zeta = \langle \zeta, \eta\rangle\xi$ whenever $\xi, \eta, \zeta \in H$.

1. PROPOSITION. *There is a one-to-one correspondence between the set of positive selfadjoint operators h on H and the set of normal semifinite weights φ on $\mathcal{B}(H)$ given by $\varphi(\xi \otimes \xi) = \|h\xi\|^2$ if $\xi \in \mathcal{D}(h)$ and $\varphi(\xi \otimes \xi) = \infty$ if $\xi \notin \mathcal{D}(h)$.*

This result essentially follows from the work of Pedersen and Takesaki [2] but can also be proved directly using a technique as in Lemma 1.4 of [1]. Moreover we have that φ is faithful if and only if h is nonsingular, and in that case the modular automorphisms are given by $\sigma_t(x) = h^{2it}xh^{-2it}$ for all $x \in \mathcal{B}(H)$ and all $t \in \mathbf{R}$.

To prove our main result we need a pair of positive selfadjoint operators with certain properties. The existence of such a pair was shown already in [3]. The proof is short and simple and we also give it here for completeness.

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