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THE DETERMINATION OF GAUSS SUMS

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1. Introduction. Almost every student with a modicum of knowledge about geometric series can show that

$$\sum_{n=0}^{p-1} e^{2\pi i n/p} = 0,$$

where p is any integer exceeding one. Suppose that we replace n by n^k in the sum, where k is an integer *greater* than one. The task of determining the sum then becomes considerably more difficult. In fact, for $k = 2$, it took Gauss several years to accomplish this. Define the Gauss sums $\mathcal{G}(k, p) = \mathcal{G}(k)$ by

$$\mathcal{G}(k) = \sum_{n} e^{2\pi i n^k/p},$$

where k is a positive integer, p is a prime with $p \equiv 1 \pmod{k}$, and Σ_n indicates that the sum on n is over an arbitrary complete residue system $(\bmod\, p)$. Closely connected with $\mathcal{G}(k)$ is the sum

$$G(\chi) = \sum_{n} \chi(n) e^{2\pi i n/p},$$

where χ is a character $(\bmod\, p)$ of order k. Both $\mathcal{G}(k)$ and $G(\chi)$ are called Gauss sums of order k and are intimately linked by the equalities

$$\mathcal{G}(k) = \sum_{n} e^{2\pi i n/p}\{1 + \chi(n) + \cdots + \chi^{k-1}(n)\} = \sum_{j=1}^{k-1} G(\chi^j). \tag{1.1}$$

The first equality in (1.1) is a simple consequence of the fact that the sequence $\{n^k\}$, $1 \leqslant n \leqslant p - 1$, runs through the set of kth power residues $(\bmod\, p)$ exactly k times.

The primary purpose of this paper is to survey the present knowledge on the values of the Gauss sums $\mathcal{G}(k)$ and $G(\chi)$, and to convey some of the principal ideas used in their determinations. We also briefly discuss more general Gauss sums.

We begin by making some elementary remarks about the values of Gauss sums. It is easily verified by direct multiplication that, for nonprincipal χ,

$$G(\chi)G(\overline{\chi}) = \chi(-1)p; \tag{1.2}$$

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