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BOOK REVIEWS

Families of meromorphic functions on compact Riemann surfaces, by Makoto Namba, Lecture Notes in Math., vol. 767, Springer-Verlag, Berlin and New York, 1979, xii + 284 pp., $16.30.

The theory of compact Riemann surfaces, or equivalently of nonsingular algebraic curves over the complex numbers, has long been a rich and rewarding field of study and remains a surprisingly lively area of current research interest. Among the problems still actively being investigated are a number involving divisors and their associated meromorphic functions, following the trail blazed by Riemann, Abel, Jacobi, and many others. Recall that a divisor on an algebraic curve M is an element of the free abelian group generated by the points of M, or in other words is a finite sum $D = \Sigma_j n_j P_j$ where $n_j \in \mathbf{Z}$ and $P_j \in M$; such a divisor is called positive or effective and written $D \geqslant 0$ if each $n_j \geqslant 0$, and the degree of the divisor is the integer $\deg D = \Sigma_j n_j$. To any meromorphic function f not identically zero on M there is associated its divisor $D(f) = \Sigma_j n_j P_j$, where n_j is the order of f at the point P_j; $n_j > 0$ if f has a zero of order n_j at P_j, $n_j < 0$ if f has a pole of order $|n_j|$ at P_j, and points at which f is of order 0 are usually not listed. If $D(f) = \Sigma_j n_j P_j$ then $\deg D = \Sigma_j n_j = 0$ and $\frac{1}{2}\Sigma_j |n_j|$ is the order of the function f; a function f of order n when viewed as a holomorphic mapping $f: M \to \mathbf{P}^1$ exhibits M as an n-sheeted branched covering of the Riemann sphere $\mathbf{P}^1$. Conversely it is traditional to associate to any divisor D the complex vector space $L(D)$ consisting of the zero function together with all those meromorphic functions f on M such that $D(f) + D \geqslant 0$; thus if $f \neq 0$ then $f \in L(\Sigma_j n_j P_j)$ if the singularities of f are at most poles of order n_j at those points P_j for which $n_j > 0$ and f has zeros of order at least $|n_j|$ at those points P_j for which $n_j < 0$. The dimension of the projective space associated to $L(D)$ is called the dimension of the divisor D and is denoted by $\dim D$, so that $\dim D = \dim_{\mathbf{C}} L(D) - 1$; considering the associated projective space rather than $L(D)$ itself really amounts to emphasizing the divisors of the functions rather than the functions themselves, since $D(f) = D(cf)$ whenever $c \in \mathbf{C}$ and $c \neq 0$. If D is a positive divisor with $\deg D = n \geqslant 2g - 1$ where g is the genus of the curve M then it follows from the Riemann-Roch theorem that $\dim D = n - g$; however if $0 < n < 2g - 1$ then $\dim D$ is a subtle and quite nontrivial function of the divisor D and the curve M.

For instance if D is a positive divisor on M with $\deg D = n$ and $\dim D \geqslant 1$ then there are at least 2 linearly independent meromorphic functions in $L(D)$, so one of them must be a nonconstant function of some order $k \leqslant n$; thus if there exists on M a divisor $D \geqslant 0$ with $\deg D = n$ and $\dim D \geqslant 1$ then M can be represented as a k-sheeted branched covering of $\mathbf{P}^1$ for some $k \leqslant n$. If M has genus $g > 1$ then $\dim D \geqslant 1$ for any divisor $D \geqslant 0$ with $\deg D \geqslant 2g - 1$; it had long been asserted in the literature that on any curve of genus g there exists a divisor $D \geqslant 0$ with $\deg D \leqslant (g + 3)/2$ and $\dim D \geqslant 1$, but the first complete proof was only given in 1960 by T. Meis, **[20]**. It