BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 2, March 1981

0002-9904/81/0000-0121/$04.00

Symmetric functions and Hall polynomials, by I. G. Macdonald, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979, viii + 180 pp., $34.95.

1. Introduction. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ be a *partition*, i.e., a (finite or infinite) sequence of nonnegative integers in decreasing order,

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant \cdots,$$

such that only finitely many of the λ_i are nonzero. The number of nonzero λ_i is called the *length* of λ, denoted $l(\lambda)$. If $\Sigma\, \lambda_i = n$, then λ is called a *partition of weight n*, denoted $|\lambda| = n$. Also write $n(\lambda) = \Sigma(i-1)\lambda_i$. Now let G be a finite abelian p-group of type λ, i.e., a direct product of cyclic groups of orders $p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_r}, \ldots,$ where p is a prime. If μ and ν are also partitions, then define $g^{\lambda}_{\mu\nu}(p)$ to be the number of subgroups H of G of type μ for which the quotient group G/H is of type ν. (Of course $g^{\lambda}_{\mu\nu}(p) = 0$ unless $|\lambda| = |\mu| + |\nu|$.) $g^{\lambda}_{\mu\nu}(p)$ is a polynomial function of p, called the *Hall polynomial*. Presented in this way, Hall polynomials appear to be of rather limited interest, of use only in dealing with enumerative properties of finite abelian groups. It is remarkable that Hall polynomials occur in many other contexts. The present book contains the first systematic account of their properties (except for the brief summary [32] which is based on some notes of Macdonald which eventually became the book under review).

2. Symmetric functions. The primary reason for the ubiquity of Hall polynomials lies in their close connection with symmetric functions. For this reason the author devotes about half of his book (Chapter I) to the theory of symmetric functions, without reference to Hall polynomials. Just this one chapter is a valuable source of information for anyone working in such fields as combinatorics, algebraic geometry, and representation theory, which frequently impinge on the theory of symmetric functions. Let us elaborate on why mathematicians in these areas should be interested in symmetric functions. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, define the *monomial symmetric function* $m_\lambda = m_\lambda(x)$ to be the formal power series in the infinite set of variables $x = (x_1, x_2, \ldots)$ given by $m_\lambda = \Sigma\, x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, summed over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of $\lambda = (\lambda_1, \lambda_2, \ldots)$. Let Λ^k be the $\mathbf{Z}$-module spanned (in fact, freely generated) by all m_λ with $|\lambda| = k$, so rank $\Lambda^k = p(k)$, the number of partitions of k. Let $\Lambda = \bigoplus_{k\geqslant 0} \Lambda^k$. Thus Λ is the free $\mathbf{Z}$-module generated by the m_λ for *all* partitions λ; and Λ has an obvious structure of a graded ring, called the *ring of symmetric functions.* (Macdonald gives a somewhat fancier definition of Λ based on inverse limits.) Chapter I is essentially concerned with the properties of certain bases for Λ^k and the transition matrices between them. This linear algebra approach toward symmetric functions is due to P. Hall [17], and is further developed in [7] and [39]. In addition to the m_λ, there are three other bases of Λ with