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THE FUNDAMENTAL THEOREM OF ALGEBRA AND COMPLEXITY THEORY[1]

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PART I

1. The main goal of this account is to show that a classical algorithm, Newton's method, with a standard modification, is a tractable method for finding a zero of a complex polynomial. Here, by "tractable" I mean that the cost of finding a zero doesn't grow exponentially with the degree, in a certain statistical sense. This result, our main theorem, gives some theoretical explanation of why certain "fast" methods of equation solving are indeed fast. Also this work has the effect of helping bring the discrete mathematics of complexity theory of computer science closer to classical calculus and geometry.

A second goal is to give the background of the various areas of mathematics, pure and applied, which motivate and give the environment for our problem. These areas are parts of (a) Algebra, the "Fundamental theorem of algebra," (b) Numerical analysis, (c) Economic equilibrium theory and (d) Complexity theory of computer science.

An interesting feature of this tractability theorem is the apparent need for use of the mathematics connected to the Bieberbach conjecture, elimination theory of algebraic geometry, and the use of integral geometry.

Before stating the main result, we note that the practice of numerical analysis for solving nonlinear equations, or systems of such, is intimately connected to variants of Newton's method; these are iterative methods and are called fast methods and generally speaking, they are fast in practice. The theory of these methods has a couple of components; one, proof of convergence and two, asymptotically, the speed of convergence. But, not usually included is the total cost of convergence.

On the other hand, there is an extensive theory of search methods of solution finding. This means that a region where a solution is known to exist is broken up into small subsets and these are tested in turn by evaluation; the process is repeated. Here it is simpler to count the number of required steps and one has a good knowledge of the global speed of convergence. But, generally speaking, these are slower methods which are not used by the practicing numerical analyst.

The contrast between the theory and practice of these methods, in my

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