6. D. G. Luenberger, *Optimization by vector space methods*, Wiley, New York, 1969.

7. G. Meinardus and D. Schwedt, *Nicht-lineare Approximationen*, Arch. Rational Mech. Anal. **17** (1964), 297–326.

8. B. N. Pschenitschny, *Notwendige Optimalitätsbegingungen*, München-Wien, 1972.

9. R. T. Rockafellar, *Duality and stability in extremum problems involving convex functions*, Pacific J. Math. **21** (1967), 167–187.

10. J. Stoer and Chr. Witzgall, *Convexity and optimization in finite dimensions*. I, Die Grundlehren der Mathematischen Wissenschaften, Band 163, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

11. J. von Neumann, *Zur Theorie der Gessllschaftsspiele*, Math. Ann. **100** (1928), 295–320.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

0002-9904/80/0000-0516/$02.25

Lectures on pseudo-differential operators: *Regularity Theorems and applications to non-elliptic problems*, by Alexander Nagel and E. M. Stein, Mathematical Notes, Princeton Univ. Press, Princeton, N. J., 1979, 159 pp., $6.75.

Pseudodifferential operators may be considered from the ontological, the teleological, or the archeological standpoint: what are they, what do they do, where do they come from? Quick answers are that they are linear operators expressed via the Fourier transform as (formal) integral operators, that they are used extensively in the study of partial differential equations, and that the direct line of descent is through singular integrals. We shall consider each point in more detail and detect as well a thread of Hegelian dialectic.

If $P = \Sigma a_\alpha(x)D^\alpha$ is a differential operator in R^n and e_ξ denotes the exponential $e_\xi(x) = \exp(ix\cdot\xi)$, then $Pe_\xi(x) = p(x, \xi)e_\xi(x)$, where $p(x, \xi) = \Sigma a_\alpha(x)\xi^\alpha$. Expressing a test function as a sum of exponentials by the Fourier inversion formula, one obtains

$$Pu(x) = \int e^{ix\cdot\xi}p(x, \xi)(\xi)\, d'\xi = \int\int e^{ix\cdot\xi}p(x, \xi)u(y)\, dy\, d'\xi, \tag{1}$$

where $d'\xi = (2\pi)^{-n}\, d\xi$. Thus the differential operator P is expressed formally as an integral operator defined by a conditionally convergent "oscillatory" integral by means of the "symbol" p. Here $p = p_r + p_{r-1} + \cdots$, where p_j is homogeneous of degree j in ξ. Thus at least locally in x one has estimates for the derivatives

$$\left|D_x^\beta D_\xi^\alpha\left(p - \sum_{k>j} p_k\right)\right| < C_{\alpha\beta j}|\xi|^{j-|\alpha|} \quad \text{if } |\xi| > 1. \tag{2}$$

These estimates may be used in conjunction with integration by parts in (1) to convert (1) into a convergent integral and verify directly that P maps $C_c^\infty(R^n)$ to $C^\infty(R^n)$. Now if Q is a second differential operator, with symbol $q = q_s + q_{s-1} + \cdots$, then the composition QP has symbol which can be calculated by Leibniz' rule:

$$q \circ p = \sum (\alpha!)^{-1}i^{-|\alpha|}D_\xi^\alpha q D_x^\alpha p. \tag{3}$$

In particular the highest order part is just the product $p_r q_s$. Note in passing that the identity operator has symbol 1.