BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

AFFINE LIE ALGEBRAS AND HECKE MODULAR FORMS

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The character of a highest weight representation of an affine Lie algebra can be written as a finite sum of products of classical Θ-functions and certain modular functions, called string functions. We find the transformation law for the string functions, which allows us to compute them explicitly in many interesting cases. Finally, we write an explicit formula for the partition function, in the simplest case $A_1^{(1)}$, and compute the string functions directly. After multiplication by the cube of the η-function, they turn out to be Hecke modular forms!

1. (See [3] or [7] for details.) Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Δ the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$. Δ_+ a set of positive roots, $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ the corresponding set of simple roots, θ the highest root. Let $(,)$ be an invariant symmetric bilinear form on $\mathfrak{g}$ normalized by $(\theta, \theta) = 2$. For $\alpha \in \mathfrak{h}^*$ with $(\alpha, \alpha) \neq 0$ define $H_\alpha \in \mathfrak{h}$ by $\beta(H_\alpha) = 2(\beta, \alpha)/(\alpha, \alpha)$ for $\beta \in \mathfrak{h}^*$. Let W be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. Denote by M the $\mathbf{Z}$-span of $W\theta$ (long roots).

Let $\mathbf{C}[t, t^{-1}]$ be the algebra of Laurent polynomials over $\mathbf{C}$ in an indeterminate t. We regard $\tilde{\mathfrak{g}} := \mathbf{C}[t, t^{-1}] \otimes_{\mathbf{C}} \mathfrak{g}$ as an (infinite-dimensional) complex Lie algebra. Define the *affine Lie algebra* $\hat{\mathfrak{g}}$ as follows. Let $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbf{C}c \oplus \mathbf{C}d$ and define the bracket by

$$[x, y]_{\hat{\mathfrak{g}}} = [x, y]_{\tilde{\mathfrak{g}}} + \operatorname{Res}_{t=0}\left(\frac{dx}{dt}, y\right)c, \ [d, x] = t\frac{dx}{dt}, \ [c, x] = 0 = [c, d],$$

for $x, y \in \tilde{\mathfrak{g}}$. The algebra $\hat{\mathfrak{g}}$ is an important example of a Kac-Moody algebra [5], [10]. Note that $\mathbf{C}c$ is the center of the algebra $\hat{\mathfrak{g}}$. The subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C}c \oplus \mathbf{C}d$ is called the *Cartan subalgebra* of $\hat{\mathfrak{g}}$. For $\alpha \in \hat{\mathfrak{h}}^*$ set $\hat{\mathfrak{g}}_\alpha = \{x \in \hat{\mathfrak{g}} \mid [h, x] = \alpha(h)x \text{ for } h \in \hat{\mathfrak{h}}\}$; then we have the root space decomposition $\hat{\mathfrak{g}} = \bigoplus \hat{\mathfrak{g}}_\alpha$.

Define a nondegenerate symmetric bilinear form $(,)$ on $\hat{\mathfrak{h}}$ by (h, h') is unchanged if $h, h' \in \mathfrak{h} \subset \hat{\mathfrak{h}}$, $(h, c) = (h, d) = 0$ for $h \in \mathfrak{h}$, $(c, c) = (d, d) = 0$, $(c, d) = 1$. We identify $\hat{\mathfrak{h}}$ with $\hat{\mathfrak{h}}^*$ by this form; then $\mathfrak{h}^*$ is identified with a subspace in $\hat{\mathfrak{h}}^*$ by $\alpha(c) = \alpha(d) = 0$ for $\alpha \in \mathfrak{h}^*$. For $\alpha \in \hat{\mathfrak{h}}^*$ set $\bar{\alpha} = \alpha|_{\mathfrak{h}^*}$ so that $\bar{\alpha} \in \mathfrak{h}^* \subset \hat{\mathfrak{h}}^*$. Define $\delta \in \hat{\mathfrak{h}}^*$ by $\delta(h) = 0$ for $h \in \mathfrak{h}$, $\delta(c) = 0$, $\delta(d) = 1$.

Received by the editors April 7, 1980.
1980 *Mathematics Subject Classification*. Primary 17B65, 10D12.