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WEIGHTS, SHARP MAXIMAL FUNCTIONS AND HARDY SPACES[1]

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A considerable development of harmonic analysis in the last few years has been centered around a function space shown in a new light, the functions of bounded mean oscillation, and the weighted inequalities for classical operators. The new techniques introduced by C. Fefferman and E. M. Stein and B. Muckenhoupt are basic in these areas. It is our purpose here to develop some of these results in a very general setting, namely that of a metric space (X, d) endowed with a doubling measure $d\mu$ and a weighted measure $d\nu = w\,d\mu$ with positive weight w. When there are a constant c and a number $q > 0$ such that if $B(x, r) = \{y \in X\colon d(x, y) \leqslant r\}$ then $\mu(B(x, rt)) \leqslant c\, t^q \mu(B(x, r))$ for all $t \geqslant 1, r > 0$ and $x \in X$ we say that μ satisfies the D_q condition and that μ is *doubling*, or $\mu \in D_\infty$, when $\mu \in D_q$ for some q. We further assume that $\mu(B(x, r))$ is a continuous function of r and that compactly supported continuous functions are dense in $L^1(d\mu)$. Because of the numerous applications of these results we feel that a detailed study is justified and a description of the new methodology needed to develop it follows.

For each $B(x, r) = B$ we define the *median* value w_B as $\sqrt{t_1 t_2}$ where $t_1 = \sup\{t > 0\colon \mu\{x \in B\colon w(x) < t\} \leqslant \mu(B)/2\}$ and $t_2 = \inf\{t > 0\colon \mu\{x \in B\colon w(x) > t\} \leqslant \mu(B)/2\}$. Then w satisfies the A_∞ condition, or $w \in A_\infty$, if $\nu(B)/\mu(B) \leqslant c w_B$. When $w \in A_\infty, w^{-1/(p-1)} \in A_\infty$ also for some $p > 1$ and there is equivalent to saying that w satisfies the usual A_p condition, or $w \in A_p$. Aside from the trivial implications the conditions A_p and D_q are independent. For A_∞ weights the following properties are obtained:

(1) $(\int_B w^r\, d\mu)/\mu(B) \sim (w_B)^r$ for $r = \beta_1 \geqslant 1$ and $r = -\beta_2 < 0$;

(2) if $B_1 \subseteq B$, then for some $\gamma_i \geqslant \beta_i$ and a constant c,

$$c^{-1}(\mu(B_1)/\mu(B))^{1+1/\gamma_2} \leqslant \nu(B_1)/\nu(B) \leqslant c(\mu(B_1)/\mu(B))^{1-1/\gamma_1};$$

(3) a strong version of the P. Jones factorization holds, to wit, if w satisfies (1) and (2) then $w = w_1 w_2$ where both w_1 and w_2 also satisfy (1) and (2) with indices $\gamma_1 - \epsilon$ and $\gamma_2 + \epsilon$. In addition $w_1(x) \geqslant c w_B$ and $w_2(x) \leqslant c w_B$ for all x in B.

The proof of (3) is too intricate to be described here but it requires the

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