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SK_1 OF p-ADIC GROUP RINGS

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If A is a Dedekind domain with quotient field K, and π a finite group, define

$$SK_1(A\pi) = \mathrm{Ker}[K_1(A\pi) \longrightarrow K_1(K\pi)].$$

We concentrate here on the case when A is a *p-ring*—the ring of integers in a finite extension of the p-adic rationals $\hat{\mathbf{Q}}_p$—and report on results which completely calculate $SK_1(A\pi)$ in this case.

The main reason for looking at $SK_1(A\pi)$ involves $SK_1(\mathbf{Z}\pi)$, shown by Wall [5] to be the torsion subgroup of the Whitehead group $\mathrm{Wh}(\pi)$ (and thus having various topological applications). The inclusions $\mathbf{Z}\pi \subseteq \hat{\mathbf{Z}}_p[\pi]$ induce a surjection

$$SK_1(\mathbf{Z}\pi) \twoheadrightarrow \sum_p SK_1(\hat{\mathbf{Z}}_p[\pi])$$

(see §1 in [3]), whose kernel is denoted $\mathrm{Cl}_1(\mathbf{Z}\pi)$. The computation of $SK_1(\mathbf{Z}\pi)$ thus splits into two parts. $\mathrm{Cl}_1(\mathbf{Z}\pi)$ can be calculated in many cases (see, e.g., [4] and [3], noting that $\mathrm{Cl}_1(\mathbf{Z}\pi) = SK_1(\mathbf{Z}\pi)$ for abelian π); but no general formula or algorithm has yet been found. The groups $SK_1(\hat{\mathbf{Z}}_p[\pi])$, on the other hand, are completely described by Theorems 1 and 2 below.

For any finite π, define

$$H_2^{ab}(\pi) = \mathrm{Im}[\textstyle\sum\{H_2(\rho)\colon \rho \subseteq \pi, \rho \text{ abelian}\} \longrightarrow H_2(\pi)].$$

If π is a p-group, the situation is particularly simple.

THEOREM 1. *For any p-ring A and p-group* π,

$$SK_1(A\pi) \cong H_2(\pi)/H_2^{ab}(\pi).$$

Note in particular that $SK_1(A\pi)$ is independent of A in this case. If $B \supseteq A$ is a totally ramified extension of p-rings, the inclusion $A\pi \subseteq B\pi$ induces an isomorphism from $SK_1(A\pi)$ to $SK_1(B\pi)$. If, on the other hand, $B \supseteq A$ is an unramified extension, it is the transfer map

$$trf\colon SK_1(B\pi) \longrightarrow SK_1(A\pi)$$

which is an isomorphism.

For arbitrary finite π, the formula is much messier. For any p-ring A and finite group π, set $n = \exp(\pi)$ and regard $\mathrm{Gal}(A\zeta_n/A)$ (ζ_n a primitive nth root of

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