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RESEARCH ANNOUNCEMENTS

LEVELS IN ALGEBRA AND TOPOLOGY

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The *level* $s(A)$ of a (commutative) ring A is the smallest natural number s such that -1 is a sum of s squares in A. (If -1 is not a sum of squares in A, we say that $s(A) = \infty$.) If A is a field, a striking result of Pfister [3] says that $s(A)$ (if finite) is always a power of 2, and indeed, all powers of 2 are possible. Knebusch and Baeza have obtained extensions of Pfister's result to semilocal rings, but little is known about levels of commutative rings in general. In [2, Problem 13], Knebusch has asked what type of integers can be the level of a ring (see also [1, p. 184]).

In this note, we announce the following.

THEOREM 1A. *For any $n \geq 1$, there exists an integral domain A with $s(A) = n$. Moreover, A can be chosen so that its field of quotients has any prescribed level $2^r \leq n$.*

A form (homogeneous polynomial) $f \in A[x_1, \ldots, x_m]$ is said to be *isotropic over A* if there exists a unimodular vector $v \in A^m$ such that $f(v) = 0$. (Otherwise, f is said to be *anisotropic over A*.) Define the *sublevel* $s'(A)$ to be the smallest integer n such that $x_1^2 + \cdots + x_{n+1}^2$ is isotropic over A. If 2 is invertible in A, it is easy to see that $s'(A)$ is equal to either $s(A)$ or $s(A) - 1$. If $s(A) \in \{1, 2, 4, 8\}$, then in fact $s'(A) = s(A)$.

THEOREM 1B. *For any $n \geq 1$, there exists an integral domain A with $s(A) = s'(A) = n$. If $n \geq 3$ is odd, there exists an integral domain B with $s(B) = n$ and $s'(B) = n - 1$.*

COROLLARY. *The pythagoras number of a ring A (i.e. the smallest integer r such that any sum of squares in A is a sum of r squares) can be any positive integer. (In fact, for the ring A in Theorem 1B, the polynomial ring $A[t]$ will have pythagoras number $n + 1$.)*

While the above results are of an algebraic nature, their proofs (at least as so far discovered) are purely topological. One uses ideas from homotopy and

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