BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 2, September 1980

BIFURCATION AND SYMMETRY BREAKING IN APPLIED MATHEMATICS

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I. Introduction. Bifurcation theory is the study of branch points in nonlinear equations, that is, of singular points of the equations where several solutions come together. It is important in applications because bifurcation phenomena typically accompany the transition to instability when a characteristic parameter passes through a critical value. To state the situation more precisely, suppose the states of a physical system are determined as solutions of a functional equation

$$G(\lambda, u) = 0 \tag{1}$$

where $\lambda \in \Lambda$ is a parameter, u is an element of a Banach space $\mathcal{E}$, and G is a mapping from $\Lambda \times \mathcal{E}$ to another Banach space $\mathcal{F}$. Let $\mathcal{S}$ be the zero set of G in $\Lambda \times \mathcal{E}$ and suppose γ is a smooth curve in $\mathcal{S}$. Then a branch point (λ_c, u_c) is a point of γ such that for any neighborhood of U of (λ_c, u_c) in $\Lambda \times \mathcal{E}$, $(U \setminus \gamma) \cap \mathcal{S} \neq \varnothing$. Some typical "bifurcation diagrams" are pictured schematically in Figure 1.

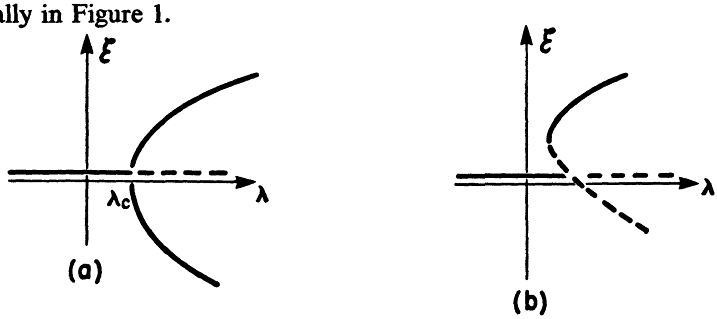

FIGURE 1.

Schematic diagram of bifurcation at $(\lambda_c, 0)$; one nontrivial branch. The vertical axis represents a Banach space $\mathcal{E}$.

Closely tied to the phenomenon of bifurcation is the property of *stability*. Suppose the solutions of (1) represent equilibrium solutions for a dynamical system which evolves according to the time dependent equations $u_t = G(\lambda, u)$. An equilibrium solution u_0 is stable if small perturbations from it remain close to u_0 as $t \to \infty$; u_0 is asymptotically stable if small perturbations decay to zero in time. When the parameter λ is varied one solution may persist but become unstable as λ crosses a critical value λ_c, and it is at such a transition point that new solutions may bifurcate from the known solution. In Figure 1 unstable solutions are represented by dashed lines.

A very simple nonlinear partial differential equation which exhibits the

Received by the editors December 15, 1979.
1980 *Mathematics Subject Classification*. Primary 35B32, 35B35; Secondary 20C99, 76E15.
[1]The preparation of this paper was supported by NSF Grant MCS 78-00415.

0002-9904/80/0000-0400/$11.50