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Set theory, by Thomas Jech, Academic Press, New York, 1978, xii + 621 pp., $53.00.

"General set theory is pretty trivial stuff really" (Halmos; see [H, p. vi]). At least, with the hindsight afforded by Cantor, Zermelo, and others, it is pretty trivial to do the following. First, write down a list of axioms about sets and membership, enunciating some "obviously true" set-theoretic principles; the most popular list today is called ZFC (the Zermelo-Fraenkel axioms with the axiom of Choice). Next, explain how, from ZFC, one may derive all of conventional mathematics, including the general theory of transfinite cardinals and ordinals.

This "trivial" part of set theory is well covered in standard texts, such as [E] or [H]. Jech's book is an introduction to the "nontrivial" part.

Now, nontrivial set theory may be roughly divided into two general areas. The first area, *classical set theory*, is a direct outgrowth of Cantor's work. Cantor set down the basic properties of cardinal numbers. In particular, he showed that if κ is a cardinal number, then 2^κ, or $\exp(\kappa)$, is a cardinal strictly larger than κ (if A is a set of size κ, 2^κ is the cardinality of the family of all subsets of A). Now starting with a cardinal κ, we may form larger cardinals $\exp(\kappa)$, $\exp_2(\kappa) = \exp(\exp(\kappa))$, $\exp_3(\kappa) = \exp(\exp_2(\kappa))$, and in fact this may be continued through the transfinite to form $\exp_\alpha(\kappa)$ for every ordinal number α. These considerations naturally led to investigations on a number of different fronts. The earliest dealt with the obvious question of whether there are any cardinals betwen κ and 2^κ. The GCH (Generalized Continuum Hypothesis) is the statement that for all infinite κ, $2^\kappa = \kappa^+$ (κ^+ is the next cardinal larger than κ). The CH is the special case, $2^{\aleph_0} = \aleph_1$, where $\aleph_0$ is the smallest infinite cardinal, or the cardinality of the set of integers, and $\aleph_1 = (\aleph_0)^+$. There were extensive investigations in the 1920s and 30s of consequences of CH, or of its negation, without yielding any insight into whether CH was really true or false. Another front is *large cardinals*, or cardinals whose size transcends those which can be produced on the basis of the ZFC axioms alone. The smallest large cardinal is an inaccessible cardinal. If κ is inaccessible, then, among other things, $\kappa > \exp_\alpha(\aleph_0)$ for any finite or countable α, or for any α of size less than κ. Measurable cardinals, which are much larger, arose naturally from measure-theoretic considerations. A third front is *infinitary combinatorics*. Once one has a "transfinite arithmetic", it is natural to consider the analogs for infinite cardinals of various questions in finite combinatorics. For example, transfinite Ramsey theory has been extensively developed by Erdös and others. A fourth front, *descriptive set theory*, grew out of a detailed study of Borel and analytic sets of real numbers, and, after Kleene's work in the 50s, was seen to be closely related to recursion theory.

The second area is *independence proofs*. Here instead of trying to prove a statement, S, from ZFC, we try to show that S is *not provable* from ZFC; equivalently, that ZFC plus the negation of S is consistent (assuming always that ZFC is consistent). S is called *independent* of ZFC iff neither S nor its negation is provable from ZFC. Such results, involving as they do the