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ASYMPTOTIC EXPANSIONS OF GAUSSIAN INTEGRALS

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Function space integrals are useful in many areas of mathematics and physics. Physical problems often give rise to function space integrals depending on a parameter and the asymptotics with respect to the parameter yield important information about the original problem. The purpose of this note is to describe the asymptotic expansions of a wide class of Gaussian function space integrals. Related work has been done by [Varadhan], [Schilder], [Pincus], [Donsker-Varadhan], and [Castro]. All asymptotic expansions previously obtained assume a nondegeneracy condition which assures that one never strays too far from the realm of Gaussian processes. Our results cover both the nondegenerate case and the degenerate case, the analysis of the latter being much more subtle. In the degenerate case, the leading asymptotic behavior is non-Gaussian.

Let P_A be a mean zero Gaussian probability measure with covariance operator A on a separable Hilbert space H. Our methods can also handle certain Banach spaces, such as $C[0, 1]$, which are important in applications. Let ψ and F be suitably bounded, real C^∞ functionals on H. We study the asymptotics of

$$J_n := \int \psi(Y/\sqrt{n}) e^{-nF(Y/\sqrt{n})} dP_A(Y) \tag{1}$$

as $n \longrightarrow \infty$.

If f and g are real-valued functions on $\mathbf{R}$, then Laplace's method tells us that the asymptotics of $\int_{\mathbf{R}} f(x)\exp(-ng(x))\,dx$ are determined by the behavior of g near its minimum points [Erdélyi, §2.4]. Formally, J_n in (1) can be written as

$$\int \psi(Y/\sqrt{n}) e^{-nF(Y/\sqrt{n})-(A^{-1}Y,Y)/2}\, dY = \int \psi(Y/\sqrt{n}) e^{-nG(Y/\sqrt{n})}\, dY, \tag{2}$$

where $G(Y) := F(Y) + (A^{-1}Y, Y)/2$. By analogy with the situation on $\mathbf{R}$, we expect the asymptotics of J_n to be determined by the behavior of G near its minimum points. The expressions in (2) are purely formal since the symbol dY

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