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THE EQUATION OF PRESCRIBED RICCI CURVATURE

BY DENNIS M. DeTURCK[1]

Introduction. In [5], J. Milnor cited "understanding the Ricci tensor $R_{ik} = \Sigma g^{jl} R_{ijkl}$" as a fundamental problem of present-day mathematics. A basic issue, then, is to determine which symmetric covariant tensors of rank two can be Ricci tensors of Riemannian metrics. The definition of Ricci curvature casts the problem of finding a metric g which realizes a given Ricci curvature R as one of solving a system of nonlinear second-order partial differential equations for g. We write these equations as

$$\mathrm{Ricc}(g) = R.$$

We note that there are the same number of equations as unknowns because g and R are both symmetric $n \times n$ matrices. However, there is a complication since any solution of $\mathrm{Ricc}(g) = R$ must also satisfy the Bianchi identity

$$\mathrm{Bian}(g, R) \equiv g^{ab}(R_{am;b} - \tfrac{1}{2}R_{ab;m}) = 0.$$

These conditions contain the unknown metric g and its first derivatives; note that the covariant derivatives of R that appear involve g via its connection.

Ultimately, one would like global results about existence, uniqueness and regularity — including topological obstructions — of metrics with prescribed Ricci tensors on manifolds. The first step, though, is to determine when one can solve the equation $\mathrm{Ricc}(g) = R$ locally, say in a neighborhood of a point x_0 in $\mathbf{R}^n$. This local problem is already nontrivial, even in the analytic case. We will exhibit examples showing that one can *not* always locally solve the equation, and we also will discuss when one can prove local solvability. Further results and details will appear in [1]. In what follows, we always assume the dimension, n, is at least 3.

1. The Bianchi identity. It is clear that a necessary condition for existence of a metric solving $\mathrm{Ricc}(g) = R$ is the existence of metrics solving the first-order (in g) equation $\mathrm{Bian}(g, R) = 0$, so we concentrate our initial efforts here. We begin with the following nonexistence result.

EXAMPLE 1.1. $R = \mathrm{diag}(x^1, \pm 1, \pm 1, \ldots, \pm 1)$ is not $\mathrm{Ricc}(g)$ for any

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