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Lectures on von Neumann algebras, by Serban Strătilă and László Zsidó, Editura Academiei, Bucureşti, România, and Abacus Press, Turnbridge Wells, Kent, England, 1979, 478 pp.

An $n \times n$ matrix algebra M_n over the complex numbers $\mathbf{C}$ aleady exhibits some of the properties of a von Neumann algebra. This algebra acts naturally on the n-dimensional vector space $\mathbf{C}^n$. It is its own double commutant in the space of linear endomorphisms on $\mathbf{C}^n$. Here the commutant of a set S of endomorphisms is the set of all endomorphisms x such that $xs = sx$ for all s in S. In addition, the algebra M_n is the dual space of the space of linear functionals on M_n generated by the evaluation of the dot product $x \to x\alpha \cdot \beta$ ($\alpha, \beta \in \mathbf{C}^n$). A distinguished functional on M_n is the trace ϕ. It takes a matrix into the sum of its diagonal elements and it is the unique functional satisfying $\phi(1) = 1$ and $\phi(xy) = \phi(yx)$ for all x, y in M_n. Then M_n is itself an inner product space with $\langle x, y \rangle = \phi(y^*x)$. Here y^* denotes the image of y under the involution equal to the transpose of the complex conjugate. The algebra M_n acts naturally on the inner product space by left multiplication. The commutant of M_n is the action of M_n by right multiplication so that M_n is its own double commutant. Again M_n is the dual space of the set of functionals on M_n defined by $x \to \langle xy, z \rangle$ ($y, z \in M_n$).

Individual elements of M_n are also important. Distinguished among these are the selfadjoint elements ($x = x^*$), the positive elements ($x = y^*y$), and the projections ($p = p^* = p^2$). A partial ordering exists for selfadjoint elements, viz. $x \geqslant y$ if $x - y$ is positive and an equivalence relation exists for projections, viz. $p \sim q$ if there is a v in M_n with $v^*v = p$, $vv^* = q$. So two projections in M_n are equivalent if and only if they have the same trace or if and only if the subspace $p(\mathbf{C}^n)$ and $q(\mathbf{C}^n)$ or the subspaces $p(M_n)$ and $q(M_n)$ (with M_n as an inner product space) have the same dimension. Furthermore, given any projections p and q in M_n then either $p \prec q$ (i.e. p is equivalent to a projection $q' \leqslant q$) or $q \prec p$.

All of these concepts find their analogue in general von Neumann algebras and illustrate some of the complexities that arise. A von Neumann algebra is a subalgebra A of the algebra of all bounded operators acting on a Hilbert space H which is closed under the natural involution of taking adjoints and is equal to its own double commutant. Unlike the case of M_n, there are several topologies. For example, the norm topology and the σ-weak topology induced by the functionals $x \to \Sigma(x\alpha_i, \alpha_i)$ where $\{\alpha_n\}$ is a sequence in H with $\Sigma\|\alpha_1\|^2 < \infty$. A subalgebra of the algebra of bounded operators on H is a von Neumann algebra if and only if it contains the identity and is σ-weakly closed. The σ-weakly continuous functionals form a Banach space in the dual of A and A is its dual space. This gives an abstract characterization of a von Neumann algebra: a C^*-algebra (i.e. a Banach algebra with involution with the property $\|x\|^2 = \|x^*x\|$) that is the dual space of a necessarily unique Banach space called the predual. It turns out that C^*-algebras are just the