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Brownian motion and classical potential theory, by Sidney C. Port and Charles J. Stone, Academic Press, New York, 1978, xii + 236 pp., $22.50.

This book does an excellent job of developing classical potential theory based on Brownian motion.

"Classical potential theory" means the theory of the Laplacian on a domain in E^n. For the purposes of this review one should think of "Brownian motion" as consisting of the entire following collection of objects: the Gauss kernel $p(t, x, y) = (2\pi t)^{-n/2} \exp(-|x - y|^2/2t)$, $t > 0$, x and y in E^n, with the corresponding semigroup $P_t f(x) = \int_{E^n} p(t, x, y) f(y)\, dy$ of operators on functions; a family $\{P^x; x \text{ in } E^n\}$ of probability measures on the space Ω of continuous paths $t \to X(t)$ from $[0, \infty)$ to E^n. The measures P^x have special properties such as (1) each P^x attributes mass 1 to the set of paths that start at the point x, (2) integrals over E^n are related to integrals over function space by formulas such as: $P_t f(x) = \int_\Omega f(X(t))\, dP^x$ and more complicated iterates of these. A probabilist would say simply that relative to each P^x the coordinate functions $X(t)$ on the sample space Ω form a time homogeneous Markov process with transition density $p(t, x, y)$ and initial position x.

Henceforth we will use the abbreviation $E^x Y$ for the integral $\int_\Omega Y\, dP^x$. Many useful aspects of the theory involve appropriately selecting Y and then developing properties of the function $x \to E^x Y$. For definiteness take $n = 3$ and define the potential of a function f or a measure μ by $Uf(x) = \int_0^\infty P_t f(x)\, dt$ and $\mu U(y) = \int_{E^3} \mu(dx)|x - y|^{-1}$, so that these are simply the familiar Newtonian potentials. On the other hand an extension of (2) above shows that $Uf(x) = E^x Y$ with $Y = \int_0^\infty f(X(t))\, dt$ and, more informally, that $\mu U(y)\, dy$ represents the expected length of time that the wandering path $t \to X(t)$ spends in the volume element dy about y if the initial position $X(0)$ is random with distribution μ. In either case the analytic object is expressed as an integral over function space.

We will give three examples to show how such representations illuminate aspects of classical potential theory. Of course all three are discussed thoroughly in the book under review. They all involve the following objects: a subset A of E^3, the (random) time T at which the path $t \to X(t)$ first meets the set A, and the place $X(T)$ at which the first meeting occurs.

(1) DIRICHLET PROBLEM. If D is a region and f is a function on its boundary, then take A to be the boundary of D and set $h(x) = E^x f(X(T))$. It is fairly obvious that on D the function h has the mean value property (any small sphere about x in D must be hit by the path before it proceeds to the boundary A, while symmetry considerations dictate that the hitting distribution on the sphere will be uniform). And if x is near the boundary and the boundary is smooth then the hitting place $X(T)$ should, with high probability, be close to x. Thus at least if f is continuous, the right boundary behavior should hold and so h solves the Dirichlet problem with boundary data f. This seems like an appealing way to start a rigorous discussion of the complete problem.