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BOOK REVIEWS

Differential geometry, Lie groups and symmetric spaces, by Sigurdur Helgason, Academic Press, New York, 1978, xvi + 628 pp., $27.00.

This review will be divided into three parts:

1. What is (or should be) a Lie group?
2. Contents of Helgason's book.
3. Comments on Helgason's book.

1. A *Lie group* is roughly speaking a group parametrized by finitely many real parameters. Before worrying about a formal definition, let us look at some basic examples. The groups in these examples are not only illustrations of the Lie theory but are, in some sense, coextensive with it.

Let **R**, **C**, **H** denote the fields of real numbers, complex numbers, and the noncommutative field of real quaternions. The letter **F** will stand for any one of these fields. Let V be an n-dimensional vector space over **F**. In case $\mathbf{F} = \mathbf{B}$, we take V to be a *right* vector space.

EXAMPLE 1. $\mathbf{R}^n$ with the usual additive structure.

EXAMPLE 2. $GL(V) =$ the group of all invertible linear transformations of V with multiplication given by composition. With a choice of a basis it may be identified with the group $GL_n(\mathbf{F})$ of $n \times n$ invertible matrices over **F**.

$SL(V)$ is the subgroup of $GL(V)$ consisting of orientation preserving Lebesgue measure preserving transformations of V. In case $\mathbf{F} = \mathbf{R}$ or **C**, $SL(V)$ can be identified with the subgroup $SL_n(\mathbf{F})$ of $GL_n(\mathbf{F})$ consisting of matrices of determinant 1. In case $\mathbf{F} = \mathbf{H}$, we may regard V as a $2n$-dimensional vector space $V^{\mathbf{C}}$ over **C** by restriction of scalars. Then $GL(V) \subseteq GL(V^{\mathbf{C}})$ and $SL(V)$ can be realized as $GL(V) \cap SL(V^{\mathbf{C}})$.

EXAMPLE 3. Let Q be a nonsingular quadratic form on V which is either bilinear or sesquilinear (with respect to standard conjugations of **C** and **H**), and either symmetric or skew. Let $O(Q)$ denote the subgroup of $GL(V)$ consisting of transformations preserving Q and $SO(Q) = O(Q) \cap SL(V)$. In detail, we then have the following seven families.

(a) $\mathbf{F} = \mathbf{R}$.

(i) A nonsingular symmetric bilinear form on V is characterized by its signature. If Q has signature h, then we shall write $O(p, q)$ for $O(Q)$, where $p + q = n, p - q = h$. Similarly, $SO(p, q)$ for $SO(Q)$.

(ii) A nonsingular skew bilinear form exists only if n is even, say $n = 2m$, in which case it is unique (up to isomorphism of quadratic spaces). One then has $O(Q) = SO(Q)$, which will also be denoted by $Sp_{2m}(\mathbf{R})$.

(b) $\mathbf{F} = \mathbf{C}$.

(i) Up to isomorphism V admits a unique nonsingular symmetric quadratic form. We write $O(n, \mathbf{C})$ for $O(Q)$ and $SO(n, \mathbf{C})$ for $SO(Q)$.

(ii) As in case (a), a nonsingular skew bilinear form exists only if n is even, say $n = 2m$, in which case it is unique up to isomorphism. Again, one has $O(Q) = SO(Q)$, which we now write as $Sp_{2m}(\mathbf{C})$.