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A CHARACTER FORMULA FOR THE DISCRETE SERIES OF A SEMISIMPLE LIE GROUP

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ABSTRACT. For a semisimple Lie group G, we provide an explicit formula for the discrete series characters θ_λ restricted to the identity component of a split Cartan subgroup, whenever the parameter lies in a so-called Borel-de Siebenthal chamber and G has both a compact Cartan subgroup and a split Cartan subgroup.

Let G be a connected semisimple Lie group with finite center. The discrete series of G is, by definition, the set of equivalence classes of irreducible unitary representations π, such that π occurs discretely in the left (or right) regular representation of G. According to Harish-Chandra [3], G has a nonempty discrete series if and only if G contains a compact Cartan subgroup. Thus we fix a compact Cartan subgroup $B \subset G$, and a maximal compact subgroup $K \subset G$ which contains B. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{b}$ be the Lie algebras of G, K, B, and $\mathfrak{g}^{\mathbf{C}}, \mathfrak{k}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}}$ their complexifications. Let $\Phi = \Phi(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ be the root system of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. A root $\alpha \in \Phi$ is called compact (respectively noncompact) if its root space lies in $\mathfrak{k}^{\mathbf{C}}$ (respectively the orthogonal complement of $\mathfrak{k}^{\mathbf{C}}$). The differentials of the characters of B form a lattice $\Lambda \subset i\mathfrak{b}^*$ ($\mathfrak{b}^*$ = dual space of $\mathfrak{b}$). The killing form induces a positive definite inner product (,) on $i\mathfrak{b}^*$. An element $\lambda \in \Lambda$ is called nonsingular if $(\lambda, \alpha) \neq 0$ for every $\alpha \in \Phi$. We set $W = W(G, B)$ = Weyl group of B in G. Equivalently, W can be described as the group generated by the reflection about the compact roots in $i\mathfrak{b}^*$.

In order to state Harish-Chandra's enumeration of the discrete series [3], we assume, without loss of generality, that G is acceptable in the sense of Harish-Chandra. Then, for each nonsingular $\lambda \in \Lambda$, there exists exactly one tempered[1] invariant eigendistribution θ_λ on G, such that

$$\theta_\lambda|_{B\cap G'} = (-1)^q \frac{\sum_{w\in W} \operatorname{sgn} w\, e^{w\lambda}}{\prod_{\alpha\in\Phi,(\alpha,\lambda)>0} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Here $q = \frac{1}{2} \dim G/K$, and G' = set of regular semisimple points in G. Every θ_λ is the character of a discrete series representation, and conversely. Moreover, $\theta_\lambda = \theta_\mu$ if and only if λ belongs to the W-orbit of μ.

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[1] A distribution θ on G is tempered if it extends to the Schwartz space of rapidly decreasing functions [3].

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