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THE NIELSEN REALIZATION PROBLEM

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If M_g is a closed, oriented 2-manifold of genus $g \geqslant 2$, then it admits many hyperbolic metrics (metrics of constant curvature -1). In special cases such a metric possesses a nontrivial group of symmetries, of isometries to itself. The group of isometries of a closed hyperbolic manifold is always finite and the only isometry isotopic to the identity is the identity itself. Thus a group of symmetries of a hyperbolic surface determines an isomorphic finite subgroup of the group of isotopy classes of diffeomorphisms of M_g. The purpose of this paper is to announce a positive solution to the *Nielsen Realization Problem* that the converse is true, i.e.; the

THEOREM 1. *Every finite subgroup of* $\pi_0 \text{Diff}\, M_g$ *can be realized as a group of isometries of some hyperbolic structure on* M_g.

For $g \geqslant 2$ the map $\text{Diff}\, M_g \longrightarrow \pi_0 \text{Diff}\, M_g$ is a homotopy equivalence, but it is unknown whether or not $\pi_0 \text{Diff}\, M_g$ can be lifted back into $\text{Diff}\, M_g$ as a subgroup. Theorem 1 solves the lifting problem for finite subgroups of $\pi_0 \text{Diff}\, M_g$.

We will call $\pi_0 \text{Diff}\, M_g$ the *modular group*[2] (Mod_g) since it is the natural generalization of the classical modular group for $g = 1$. Another common name is the *mapping class group.*[1] It is naturally isomorphic to the group of outer automorphisms of $\pi_1 M_g$. Mod_g acts on the Teichmüller space (T_g) of hyperbolic metrics on M_g (where two are considered equivalent if there is an isometry, isotopic to the identity, between them). T_g is homeomorphic to a $(6g - 6)$-dimensional ball and the action of Mod_g on T_g is properly discontinuous and, except for an element of order two in genus two, faithful. Theorem 1 is equivalent to

THEOREM 2. *Every finite subgroup of* Mod_g *acting on* T_g *has a fixed point.*

For $g = 1$, T_g (for flat metrics of area 1) is the upper half plane, $\text{Mod}_g \approx GL(2, \mathbf{Z})$ acts by linear fractional transformations, and Theorem 2 is classical. In the higher genus case, Theorems 1 and 2 were known for G cyclic (Nielsen [4]), solvable (Fenchel [1]), and in many other special cases (see Zieschang [6]). Kravetz [2] gave a proof of Theorem 2, but it was based on the false belief that T_g, equipped with Teichmüller's metric, has negative curvature (see Masur [3]).

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[2] We make no restriction to the orientation-preserving case, however.