BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 3, May 1980

# AN IDENTITY WITH APPLICATIONS TO HARMONIC MEASURE

BY DAVID S. JERISON[1] AND CARLOS E. KENIG[2]

In this note we use an elementary integral formula to give a new short proof of the theorem of B. E. J. Dahlberg [2] on the mutual absolute continuity of harmonic and surface measure on bounded Lipschitz domains. (See [4] for the relevant definitions.) The formula also provides a new proof of the so-called reverse Holder inequality (see [2]), and $L^p$-estimates for the Dirichlet problem (see [3]). Furthermore, we are able to treat domains whose boundaries are worse than Lipschitz. In particular, we show that if $D \subset R^n$ is given locally (in the appropriate sense) by the graph of a function $\phi$ with $\nabla\phi \in L^p$, $p > n - 1$, and $D$ is regular for the Dirichlet problem, then harmonic measure and surface measure on $\partial D$ are mutually absolutely continuous. The hypothesis of regularity is superfluous when $n = 3$, or under the stronger assumption $\nabla\phi \in \mathrm{BMO}$, but not necessarily otherwise.

We wish to thank Professor E. M. Stein for many helpful conversations.

THEOREM 1. *Let $D$ be a bounded Lipschitz domain in $\mathbf{R}^n$. Let $\omega$ be harmonic measure of $\partial D$ with respect to $X_0 \in D$. Then*

(a) *$\omega$ and $\sigma$ are mutually absolutely continuous.*

(b) *Let $k(Q) = d\omega/d\sigma$. Then $k \in L^2(d\sigma)$; moreover, for every surface ball $\Delta \subset \partial D$,*

$$(*) \qquad \left(\frac{1}{\sigma(\Delta)} \int_\Delta k^2(Q)\, d\sigma(Q)\right)^{1/2} \leqslant \frac{c}{\sigma(\Delta)} \int_\Delta k(Q) d\sigma(Q).$$

To establish our theorem, we need

LEMMA 1 (THE MAIN IDENTITY). *Assume that $D$ is a bounded $C^\infty$ domain in $\mathbf{R}^n$, containing 0. Let $\omega$ be harmonic measure of $\partial D$ at 0. Then if $k = d\omega/d\sigma$,*

$$\frac{1}{\omega_n} \int_{\partial D} \frac{k(Q)}{|Q|^{n-2}} d\sigma(Q) = \int_{\partial D} k^2(Q) \langle Q, N_Q\rangle\, d\sigma(Q),$$

*where $N_Q$ is to outer unit normal, and $\omega_n$ the surface area of $S^{(n-1)}$.*

---

Received by the editors November 16, 1979.
*AMS (MOS) subject classifications* (1970). Primary 31BXX, 31B05, 31B25.
[1] The first author was partly supported by an NSF graduate fellowship.
[2] The second author was supported in part by an NSF grant.


0002-9904/80/0000-0205/$02.25