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# NUMBER THEORETICAL DEVELOPMENTS ARISING FROM THE SIEGEL FORMULA

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**1. Introduction.** Siegel Formula is the name Weil gave to an equality which relates a theta series with an Eisenstein series [**16**]. The original result of Siegel is quite arithmetic in nature, with special cases yielding for example Fermat's theorem that every prime $p \equiv 1 \pmod 4$ is expressible as a sum of two squares in essentially one way; Jacobi's theorem on the number of ways of writing an integer as a sum of four squares–namely $8\, \Sigma_{2d+1|n}\, (2d+1)$ for $n$ odd or $24\, \Sigma_{2d+1|n}\, (2d+1)$ for $n$ even; and also Dirichlet's class number formula. In the more arithmetically accessible case, Siegel's theorem can be stated as follows:

Let $h$ be a positive definite quadratic form in $m$ variables, with integer coefficients, write

$$d(t) = \#\{x \in \mathbf{Z}^m | h(x) = t\},$$

$$d_p(t) = \operatorname*{limit}_{t\to\infty} \frac{\#\{x \in (\mathbf{Z}/p^r\mathbf{Z})^m | h(x) \equiv t \pmod{p^r}\}}{p^{r(m-1)}},$$

$$d_\infty(t) = \operatorname*{limit}_{D\to\{t\}} \frac{\text{volume } h^{-1}(D)}{\text{volume } (D)},$$

where $D$ runs through the compact neighbourhoods of $t$ and $h\colon \mathbf{R}^m \to \mathbf{R}$. Then we have $d(t) = d_\infty(t)\Pi_p\, d_p(t)$. (At least this formulation is correct when $m > 2$ and the genus of $h$ has only one class. See §6 for a discussion of genus, class of $h$.)

In general Siegel characterises his result as having the same quantitative relationship to the Hasse-Minkowski theorem, as the Jacobi Theorem mentioned above has to Lagrange's result that every integer is a sum of four squares.

One can ask also for the number of representations of an $n \times n$ integral, symmetric, positive definite matrix $T$ by a given $m \times m$ one $S$, and once more Siegel has a similar result. Moreover, Siegel has generalized his theorem to $T$ indefinite and where the coefficients lie in an algebraic number field only now the definition of the densities $d_p(t)$ are much more involved, with no ready arithmetic interpretation. The proofs consist of constructing an Eisenstein series $E(\tau)$ which behaves like the generating function $f(\tau)$ for our Diophantine problem:

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