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THE ALGEBRAIST'S UPPER HALF-PLANE

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Introduction. The purpose of this article is to introduce the general mathematical community to some recent developments in algebraic geometry and nonarchimedean analysis. Let $r = p^n, p$ a rational prime. Then these developments center around the beginnings of an "arithmetic" theory of the polynomial ring $\mathbf{F}_r[T]$ over the finite field of r elements. The goal of this theory is to use nonarchimedean analysis to do for $\mathbf{F}_r[T]$ what classical analysis does for $\mathbf{Z}$. The theory allows us to find *direct* analogues of many of the classical functions of arithmetic interest in a situation that, at first glance, seems as nonclassical as possible. In the process much will be learned about the polynomials. Much also will be learned about the unique properties of $\mathbf{Z}$ and the classical functions.

One of the exciting aspects of the theory is its great generality. Indeed, we could replace $\mathbf{F}_r[T]$ with much more general affine rings of curves over finite fields. More precisely, if C is a projective, smooth curve over $\mathbf{F}_r$, ∞ a rational point and A the functions regular away from ∞, then we may use A instead of $\mathbf{F}_r[T]$. Thus one can, so to speak, get a sense of what analysis might have been forced to if $\mathbf{Z}$ were not a unique factorization domain. Such observations can only come in the present setting since $\mathbf{Q}$ is the only totally real (i.e., all Galois conjugates contained in $\mathbf{R}$) field with a *unique* absolute-value. We have chosen to stick to the polynomials in order to keep the exposition as simple as possible. The jump from the polynomials to more general rings is not terribly large and most essential features appear for $\mathbf{F}_r[T]$.

Another exciting aspect is that we begin to see how a given 'arithmetic' situation generates an associated harmonic analysis. As classical harmonic analysis is based on the integers, the one developed here is based on $\mathbf{F}_r[T]$. In contrast to classical harmonic analysis which is multiplicative, i.e., based on the exponential function, the one here is based on *addition*.

Throughout the paper we compare the theory here with the classical one. In this fashion we hope the reader may speedily develop a feel for the subject.

One of the most surprising (and hotly contested) aspects of classical analysis is its harmonic analysis. This centered around the possibility of expanding an *arbitrary* singly-periodic function in terms of sines and cosines. Since sines and cosines are easily expressed in terms of the exponential function, $e^{(z)}$, the central role of this function is apparent.

Viewed on the complex plane, $e^{(z)}$ has the following very well-known properties: It is never zero, takes addition to multiplication, is invariant under $z \mapsto z + 2\pi i$ and, finally, it is its own derivative. As a consequence, $e^{(z)}$ gives

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