BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0024/$02.00

Multiple-conclusion logic, by D. J. Shoesmith and T. J. Smiley, Cambridge Univ. Press, Cambridge, 1978, xiii + 396 pp., $35.00.

The subject of this book is meta-meta-mathematics; it is related to meta-mathematics as the latter is to the rest of mathematics. The aim of mathematics is to describe interesting objects and phenomena in precise terms, expressing their basic properties as axioms, and to deduce interesting theorems from these axioms. Meta-mathematics is the part of mathematics where the phenomenon to be described is the process of mathematical deduction. Its basic concept is the relation of logical consequence, which holds between a set X of statements and a statement A (written $X \vdash A$ by Shoesmith and Smiley although $X \vDash A$ is more common) when A is true in every conceivable situation where all the members of X are true. Its results, such as Gödel's completeness theorem, apply to all the various axiom systems of mathematics regardless of their meaning or purpose.

The definition of $\vdash$ is vague in that we have not specified what situations are conceivable. Traditionally, one regards the meanings of the logical connectives (not, and, or, . . .), the quantifiers (for all, for some), and equality as fixed, so that a situation where one of these has a nonstandard interpretation is considered inconceivable, but the meanings of other expressions are permitted to vary. The resulting $\vdash$ is the consequence relation of the classical first-order predicate calculus. If, in addition, we fix the meaning of "natural number", we obtain (the consequence relation of) a stronger logical system called ω-logic. If we fix the meaning of "set" we obtain second-order logic (which is stronger than ω-logic since the Peano axioms provide a characterization of the natural numbers). In the other direction, if we unfix the meanings of the quantifiers and equality, we obtain propositional logic.

Other consequence relations are produced by varying the meaning of "true". The most important of these are the constructive logics such as intuitionism, where "true" is taken to be synonymous with "proved" or with "provable". There are also many-valued propositional logics, where one has a set of two or more (usually more) truth values, some of which are designated as "true", and for each logical connective one has a corresponding operation on truth values. A simple but useful example has as truth values the four ordered pairs of ordinary truth values (true and false), with only $\langle$true, true$\rangle$ designated, and with the connectives operating componentwise. This example is rather special in that is has the same consequence relation as classical propositional logic.

Shoesmith and Smiley develop a theory of consequence relatons in general, intended to be applicable to all the preceding examples (and others). They work, however, with consequence relations, $\vdash$, both of whose arguments, not just the left one, are sets of statements. If $X \vdash Y$ were interpreted in the obvious way as "Any conceivable situation making all the statements in X true also makes all the statements in Y true", this would reduce to the single-conclusion concept of $\vdash$ since it is equivalent to "$X \vdash A$ for all $A \in Y$". In order to get an interplay between the elements of Y similar to what