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DISPLACEMENT RANKS OF A MATRIX[1]

BY T. KAILATH, S.-Y. KUNG AND M. MORF

The solution of many problems in physics and engineering reduces ultimately to the solution of linear equations of the form $Ra = m$, where R and m are given $N \times N$ and $N \times 1$ matrices and a is to be determined. Here our concern is with the fact that it generally takes $O(N^3)$ computations (one computation being the multiplication of two real numbers) to do this, and this might be a substantial burden if N is large or if the problem has to be repeated with different R and m. For such reasons, one often seeks to impose more structure on the matrices R. In many problems we have an underlying stationarity or homogeneity (invariance under displacements in time or space) property that often leads to the matrix R being *Toeplitz* (i.e., with elements of the form R_{i-j}). Now it is known that Toeplitz matrices can be inverted with $O(N^2)$ (or even $O(N \log^2 N)$) multiplications, which can be considerable simplification. However, even if the physical problem has an underlying stationarity property, it still happens that in the course of the analysis the coefficient matrix R turns out to be non-Toeplitz, though in some sense close to Toeplitz. For example R may be the inverse of a Toeplitz matrix, or the product of two rectangular Toeplitz matrices (as arises often in least-squares theory), or an asymptotically Toeplitz matrix ($R_{ij} \longrightarrow R_{i-j}$ as $i, j \longrightarrow \infty$). It seems unreasonable that equations with such non-Toeplitz matrices should require $O(N^3)$ operations for their solution, but this question does not seem to have been systematically explored before.

Motivated by a number of related results on the solution of certain nonlinear (Riccati- and Chandrasekhar-type) differential equations arising in some least-squares estimation problems ([1]-[3]), we have been able to provide some answers to the above question and also obtain some extensions. Roughly speaking, with an $N \times N$ matrix R we are able to associate an integer α, $1 \leqslant \alpha \leqslant N$, that seems to provide a nice measure of how close R is to being Toeplitz; moreover, we have shown that a matrix with index α can be inverted with (about) α times as much computations as required for a Toeplitz matrix.

To make these statements more precise, we introduce two so-called *displacement ranks* of a matrix.

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