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DIVISION ALGEBRAS OF DEGREE 4 AND 8 WITH INVOLUTION

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ABSTRACT. Examples are given of division algebras with involution $(*)$ of the first kind, one of degree 8 which is not a tensor product of quaternion subalgebras, the other of degree 4 which is not a tensor product of $(*)$-invariant quaternion subalgebras.

Suppose D is a division algebra with center F, and $[D: F] < \infty$. Then $[D: F] = n^2$ for suitable n; n is the *degree* of D, and D is a *quaternion F-algebra* when $\deg(D) = 2$. We further assume D has characteristic $\neq 2$, and has an involution $(*)$ of the first kind, i.e. $(*)$ is an anti-automorphism of degree 2 which fixes F. This situation is treated in depth in [1, Chapter 10], and it arises if and only if D has exponent 2 in the Brauer group, i.e. $D \otimes D^{\mathrm{op}} \approx M_{n^2}(F)$, the algebra of $n^2 \times n^2$ matrices over F. Thus, in this case, the degree of D is a power of 2. Until now, the only known such algebras were tensor products of quaternion F-subalgebras.

QUESTION 1. Is D necessarily a tensor product of quaternion F-subalgebras?

QUESTION 2. Is D necessarily a tensor product of $(*)$-invariant quaternion F-subalgebras?

Question 1 dates back about 60 years; Albert [1] showed it is true when $\deg(D) \leqslant 4$. The main object of this paper is to give a counterexample for degree 8. Also, we shall give a counterexample to Question 2 for degree 4, which is clearly sharp. (Incidentally, for *symplectic* involutions, question 2 has no counterexample of degree 4, cf. [3, Theorem B].) Our counterexample makes the following result of Tignol [4] sharp: If $\deg(D) = 8$ then $M_2(D)$ is a tensor product of quaternion subalgebras. A more detailed description of our methods will appear in the *Israel Journal of Mathematics.*

The main idea is to use the generic abelian crossed products of [2], modified slightly to account for the presence of an involution. Suppose R is an abelian crossed product, i.e. D has a maximal subfield K Galois over F, having Galois group $G = \langle \sigma_1 \rangle \oplus \cdots \oplus \langle \sigma_q \rangle$, a direct sum of cyclic groups, and for our purposes we assume that σ_i has order 2. Then, choosing z_i such that $\sigma_i(x) = z_i x z_i^{-1}$ for all x in K, we define $u_{ij} = z_i z_j z_i^{-1} z_j^{-1}$ and $b_i = z_i^2$, elements of K. [2, Lemma 1.2] gives the following conditions for all i (where $N_i(x) = x\sigma_i(x)$ by definition).

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