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THE L^2-INDEX THEOREM FOR HOMOGENEOUS SPACES

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The geometric realization of the irreducible square integrable representations for semisimple Lie groups (cf. [3], [6]) and also for nilpotent Lie groups [5] suggests that, as a general phenomenon, such representations should appear as L^2-kernels of invariant elliptic operators. One basic problem in this respect is to decide when such a kernel is nonzero. In the compact case the basic tool for this, used in the Borel-Weil-Bott approach, is the Hirzebruch-Riemann-Roch theorem. In the noncompact case one needs an analogue of the index theorem of Atiyah-Singer [2] for noncompact manifolds. When G possesses a discrete cocompact subgroup, the L^2-index theorem for covering spaces of [1] and [7] provides the required analogue. Our purpose here is to give a general index theorem for homogeneous spaces of arbitrary connected unimodular Lie groups, essentially based on the index theorem for foliations [4].

So let G be a connected unimodular Lie group, and let H be a closed subgroup of G which contains the center Z of G and such that H/Z is compact. Let χ be a character of Z, and let E, F be finite-dimensional unitary representations of H whose restrictions to Z are given by χ. Denote by $\mathbf{E}$, $\mathbf{F}$ the corresponding (invariant) induced bundles on the homogeneous space $M = G/H$, and let D be an invariant elliptic differential operator from $\mathbf{E}$ to $\mathbf{F}$. The representation of G in the kernel of D in $L^2(M, \mathbf{E})$ is square integrable modulo the center of G (see [4]), though not necessarily irreducible. Its formal degree deg(Ker D) (as defined in [4]) is always finite, so that the analytical index of D can be defined as

$$\operatorname{Ind}(D) = \deg(\operatorname{Ker} D) - \deg(\operatorname{Ker} D^*).$$

We now describe the topological index of D. Let $\mathfrak{M}$ be an Ad H invariant supplement for Lie (H) in Lie (G). We can assume, dividing by Ker χ, that H is compact. The principal symbol of D defines an element σ_D of $K_H(\mathfrak{M}^*)$, the equivariant K-theory with compact support of the dual vector space $\mathfrak{M}^*$ of $\mathfrak{M}$. Using the Thom isomorphism at the level of the rational cohomology of the classifying space for H, one gets a natural map τ from $K_H(\mathfrak{M}^*)$ to the completion $(R(H) \otimes \mathbf{Q})^\wedge$ of the representation ring of H. Let then $H_G^*(M, \mathbf{R})$ be the cohomology ring of G-invariant differential forms on the homogeneous space M.

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