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# RESEARCH ANNOUNCEMENTS

## FINITENESS THEOREMS FOR POLYCYCLIC GROUPS

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**Introduction.** A group $G$ is *polycyclic* if it is built up from the identity by finitely many successive extensions with cyclic groups, or equivalently if it is isomorphic to a soluble group of matrices over $\mathbf{Z}$ (not obvious!). The second definition makes it clear that the normal subgroups of finite index in $G$ intersect in 1, so one may hope that the finite quotient groups of $G$ will carry a lot of information about the structure of $G$. The first main result says that in fact they "almost" determine $G$ up to isomorphism, i.e. they do so up to finitely many possibilities. (Examples show that there really are finitely many possibilities, not just one.) The second main result is a sort of "concrete" analogue of this: if $G$ is contained in $GL_n(\mathbf{Z})$, then there are only finitely many possibilities up to conjugacy in $GL_n(\mathbf{Z})$ for subgroups $H$ in $GL_n(\mathbf{Z})$ such that $H$ is "conjugate to $G$ modulo $m$" for all nonzero integers $m$. This is related to classical results in arithmetic, like the fact that there are only finitely many inequivalent integral quadratic forms with given determinant, and the Hasse-Minkowski Theorem.

**Results.** Denote by $\mathfrak{F}(G)$ the set of isomorphism classes of finite quotients of a group $G$, and by $\hat{G}$ the profinite completion of $G$. For polycyclic-by-finite groups $G$ and $H$, $\mathfrak{F}(G) = \mathfrak{F}(H)$ if and only if $\hat{G} \cong \hat{H}$; if this holds we say that $G$ and $H$ belong to the same $\hat{}$ -class.

THEOREM 1. *Every $\hat{}$ -class of polycyclic-by-finite groups is the union of finitely many isomorphism classes.*

A major ingredient in the proof of this is a result about arithmetic groups. Let $G$ be an algebraic matrix group defined over $\mathbf{Q}$, and denote by $\pi_m$: $G(\mathbf{Z}) \longrightarrow G(\mathbf{Z}/m\mathbf{Z})$ the canonical map. For subgroups $X$ and $Y$ of $G(\mathbf{Z})$, say $X \sim_G Y$ if for every $m \neq 0$, $X\pi_m$ and $Y\pi_m$ are conjugate in $G(\mathbf{Z}/m\mathbf{Z})$.

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