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PROBLEMS IN HARMONIC ANALYSIS RELATED TO CURVATURE

BY ELIAS M. STEIN AND STEPHEN WAINGER[1]

PART I–SURVEY

1. Preface. Our point of departure for this paper is the standard real variable theorem of Lebesgue on the differentiation of integrals of functions of several variables. Lebesgue's theorem asserts that for any locally integrable function f defined on R^n

$$\lim_{\varepsilon\to 0} \frac{1}{|B(\varepsilon, x)|} \int_{B(\varepsilon,x)} f(y)\,dy = f(x) \quad \text{a.e.},$$

for certain types of n-dimensional sets $B(\varepsilon, x)$ shrinking to x as $\varepsilon \to 0$. ($|B(\varepsilon, x)|$ is of course the Lebesgue measure of $B(\varepsilon, x)$.) Some standard examples of sets $B(\varepsilon, x)$ are balls with center x and radius ε, and cubes with center x and diameter ε.

The problem of considering other sets besides balls and cubes for $B(\varepsilon, x)$ received much attention in the 1930's. (See for example Buseman and Feller **[1934]**.) This subject again seems to be attracting interest.

One of the two main goals of this paper is an exposition of recent developments in which $B(\varepsilon, x)$ is replaced by a lower dimensional set.[2] Specifically we shall examine two cases:

(i) $B(\varepsilon, x)$ is replaced by a sphere with center at x and radius ε,

(ii) $B(\varepsilon, x)$ is replaced by a piece of a curve emanating from x.

Thus we ask, does

$$\lim_{\varepsilon\to 0} \int_{|y|=1} f(x-\varepsilon y)\,d\sigma(y) = f(x) \quad \text{a.e.?} \tag{1}$$

Here $d\sigma$ is the rotationally invariant measure on the sphere $|y| = 1$ having total mass 1. For continuous f the answer is clearly yes. For discontinuous f it is not even clear that $\int f(x - \varepsilon y)\,d\sigma(y)$ is well defined for all small ε for almost every x. In one dimension the answer to 1 is clearly no, even if f is restricted to lie in the class of bounded functions. No positive results for (1) are true in any dimension for all locally integrable f; however the answer is yes if we restrict our attention to f's which are locally in an appropriate L^p class and $n \geqslant 3$. Specifically, we have Theorem A.

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[2] References to some other recent results of a different nature will be found in part III.