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HILL'S SURFACES AND THEIR THETA FUNCTIONS[1]

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Preface. In this paper we continue the investigations begun in McKean-Trubowitz [1976] of infinite-genus hyperelliptic Riemann surfaces S which are constructed from the spectrum of a Hill's operator. Let q be a real infinitely differentiable function of $0 \leqslant \xi < 1$ of period 1. The Hill's operator is $Q = -d^2/d\xi^2 + q(\xi)$. The periodic and antiperiodic eigenfunctions of Q determine an infinite spectrum $\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \cdots \uparrow\infty$ of simple or double eigenvalues. S is formed by cutting two copies of the number sphere along the so-called intervals of instability marked off by such pairs of simple eigenvalues $\lambda_{2n-1} < \lambda_{2n}$ as may occur. S is a hyperelliptic surface of genus g ($\leqslant$ infinity) equal to the number of such pairs. The purpose of this paper is to develop some of the function theory of S in the case g = infinity with special attention to differentials of the first kind, the Jacobian variety, and the Riemann theta function. McKean-Trubowitz [1976] introduced a Hilbert space of differentials of the first kind closely connected with the interpolation of certain classes of entire functions, defined the Jacobi map for divisors in "real position", and constructed the "real part" of the Jacobian variety. The present paper studies more refined Hilbert spaces of differentials; one such space of particular importance is populated by differentials with precisely "$2 \times (g = \text{infinity}) - 2$" roots, just as in the classical case. The associated (infinite-dimensional) Jacobian variety and its theta function are also introduced. The basic properties of the latter include a variant of the Riemann vanishing theorem. A theta function formula of Baker [1897] and Its-Matveev [1975] is adapted to the present case and used to express the solution of the celebrated Korteweg-deVries equation with periodic initial data. Along the way, we prove period relations, derive an infinite-dimensional analogue of Jacobi's identity for the theta function, embed S in its Jacobian variety, and prove the easy half of Abel's theorem.

To the best of our knowledge the only previous work on transcendental hyperelliptic function theory is that of Hornich [1933], [1935], [1939] and Myrberg [1943], [1945]. These papers contain some discussion of square-summable differentials of the first kind and their period relations. The

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