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SOME APPLICATIONS OF MODULE THEORY TO FUNCTOR CATEGORIES

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Introduction. The notion of an additive category was abstracted from the example of all modules over a ring, a very large category. However with tongue firmly in cheek, one can define a ring with identity (all rings will have identity) as an additive category with just one object. Years ago, the notion of a Morita context was expounded with a certain amount of labour. A Morita context turned out to be an additive category with two objects. It is not inconceivable, then, that someday additive categories with three objects will emerge, the jump from three to infinity will be made, and additive categories will be rediscovered from the point of view of the small examples instead of the big ones. (Heaven knows what they will be called.)

I wish to indicate how the observation that a ring R is an additive category with one object can be used for purposes other than to boggle the student of algebra. First, an R-module, from this point of view, is just an additive functor from R to the category Ab of abelian groups, and an R-module homomorphism is a natural transformation between two such functors. Thus, if $\mathcal{C}$ is a small additive category, or what we shall refer to more briefly as a *ringoid*, then a $\mathcal{C}$*-module* is a covariant additive functor M: $\mathcal{C} \to$ Ab, and the category of all such is denoted Mod $\mathcal{C}$. (Actually what we have defined is a *left* $\mathcal{C}$-module, a *right* $\mathcal{C}$-module being an object of Mod $\mathcal{C}^{\mathrm{op}}$.) Now frequently, when such a category arises in the literature, it is pointed out that it is an abelian category, that it has exact direct limits, that it has a set of generators, that it has enough projectives and the injectives, and so on. What needs to be stressed is that there is virtually nothing which one can do in categories of modules over (not necessarily commutative) rings, which can't be done in categories of modules over ringoids.

First, let us consider the building block of the category Mod R, namely R considered as a module over itself. In the more general situation, there is a whole family of building blocks, one for each object of $\mathcal{C}$, namely the representable modules (functors) $\mathcal{C}(p, \ )$. The additive Yoneda lemma states that there is an isomorphism of abelian groups

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(p, \ ), M) \simeq M(p) \tag{1}$$

which is natural not only as functors of the $\mathcal{C}$-module M, but also as functors of the variable p. What is being generalized here, of course, is the familiar natural isomorphism $\mathrm{Hom}_R(R, M) \simeq M$ of left R-modules. The

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