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PROPAGATION, REFLECTION, AND DIFFRACTION OF SINGULARITIES OF SOLUTIONS TO WAVE EQUATIONS

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0. Introduction. This paper will survey recent progress in understanding the propagation of singularities of solutions to linear partial differential equations $Pu = f$, particularly hyperbolic equations, such as the wave equation $(\partial^2/\partial t^2 - \Delta)u = f$. Theorems describing this behavior, for general initial data, probably began with Lax [21] and Courant and Lax [6], although work on the problem dates back further. The method of analysis, known as geometrical optics, was used by Sommerfeld and Runge [44] and Birkhoff [2] in an effort to construct approximate solutions to the wave equation. This method was forged into a powerful tool, the theory of Fourier integral operators, by Hörmander [15], [16] and applied to get very general global results on propagation of singularities in [16] and [8].

In order to give a precise statement of Hörmander's theorem on propagation of singularities, we need to define the wave front set of a distribution, denoted $\mathrm{WF}(u)$, where $u \in \mathcal{D}'(\Omega)$ is a distribution on some domain $\Omega \subset \mathbf{R}^n$. $\mathrm{WF}(u)$ was introduced by Hörmander [15], based on Sato's notion of S. S. u [42]. $\mathrm{WF}(u)$ will be a subset of $T^*(\Omega) \approx \Omega \times \mathbf{R}^n$. One way to give the definition is to say $(x_0, \xi_0) \notin \mathrm{WF}(u)$ provided there is a $\varphi \in C_0^\infty(\Omega)$, $\varphi = 1$ near x_0, such that $(\varphi u)\hat{}(\xi)$ is rapidly decreasing as $|\xi| \to \infty$ for ξ in some open cone Γ containing ξ_0. An equivalent definition, using pseudo differential operators, will be given in §1. It turns out that the projection $T^*(\Omega) \to \Omega$ maps $\mathrm{WF}(u)$ onto the singular support of u (sing supp u), so $\mathrm{WF}(u)$ provides finer information than sing supp u.

Now suppose $Pu = f$ in Ω. We suppose P is a differential operator, or more generally a pseudo differential operator of order m, whose principal symbol $p_m(x, \xi)$, homogeneous of degree m in ξ, is *real valued*. Let $q(x, \xi) = |\xi|^{1-m} p_m(x, \xi)$, and consider the Hamiltonian vector field on $T^*(\Omega)$:

$$H_q = \sum_{j=1}^{n} \left(\frac{\partial q}{\partial x_j} \frac{\partial}{\partial \xi_j} - \frac{\partial q}{\partial \xi_j} \frac{\partial}{\partial x_j} \right).$$

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