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UNIFORM EQUIVALENCE BETWEEN BANACH SPACES

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It is a well-known result of Kadec that every two separable infinite dimensional Banach spaces are homeomorphic. Also in large classes of nonseparable Banach spaces (perhaps all) the density character of a Banach space is its only topological invariant (see the book [2] for details). The situation changes considerably if we consider uniform homeomorphisms. Several results are known which prove the nonexistence of uniform homeomorphisms between certain Banach spaces of the same density character. As a matter of fact, the following problem was raised by many mathematicians: Do there exist two nonisomorphic Banach spaces which are uniformly homeomorphic? (i.e. does the uniform structure of a Banach space determine its linear structure?) For a recent survey of results related to this problem see [3].

While studying the question of existence of nonlinear liftings, we found in a surprisingly simple manner an example which answers this problem. Let Γ be a set of the cardinality of the continuum. Then $c_0(\Gamma)$ is Lipschitz equivalent to a certain closed subspace X of l_∞ (i.e. there is a map T from $c_0(\Gamma)$ onto X so that T and T^{-1} satisfy a Lipschitz condition). Since there is no sequence of continuous linear functionals which separate the points in $c_0(\Gamma)$, this space is not isomorphic to a subspace of l_∞.

Let $U \supset V$ be Banach spaces and let $\varphi: U \longrightarrow U/V$ be the quotient map. We say that φ admits a Lipschitz (resp. uniformly continuous) lifting if there is a Lipschitz (resp. uniformly continuous) map $\psi: U/V \longrightarrow U$ so that $\varphi\psi$ is the identity of U/V. If such a lifting exists then U is Lipschitz (respectively uniformly) homeomorphic to the direct sum $V \oplus U/V$. A suitable map T which establishes the homeomorphism is $Tu = (u - \psi\varphi(u), \varphi(u))$.

Let $\{N_\gamma\}_{\gamma\in\Gamma}$ be a collection of subsets of the integers N so that each N_γ is infinite and $N_\gamma \cap N_{\gamma'}$ is finite for $\gamma \neq \gamma'$. Let X be the closed linear subspace of l_∞ spanned by c_0 and the characteristic functions χ_γ of N_γ, $\gamma \in \Gamma$. Clearly X/c_0 is isometric to $c_0(\Gamma)$ with $\varphi\chi_\gamma$ corresponding to the natural unit vectors e_γ of $c_0(\Gamma)$. The map φ admits a Lipschitz lifting and thus X is Lipschitz equivalent of $c_0(\Gamma) \oplus c_0 \approx c_0(\Gamma)$. Indeed, let

$$y = \sum_{n=1}^{t} a_n e_{\gamma_n} - \sum_{m=1}^{s} b_m e_{\gamma'_m}$$

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